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MATHEMATICAL IMPLEMENTATION OF BOUNDARY ELEMENT METHOD

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ABSTRACT

The Boundary Element Method is defined as type of a numerical method that approximates solutions of boundary value problems. The most important aspect in which the Boundary Element Method distinguishes itself from other numerical methods is the fact that only the boundary of a domain needs to be discretized. In many other numerical methods, such as the finite element method, finite differences or the finite volume method, in addition to the boundary, the interior of the domain also needs to be discretized. As a consequence of the boundary discretization, the Boundary Element Method is a suitable method for problems on external domains, or domains that have a free or moving boundary. Also problems in which singularities or discontinuities occur can be handled efficiently by the Boundary Element Method. Another advantage of the Boundary Element Method is that variables and their derivatives, for instance temperature and its flux, are computed with the same degree of accuracy.

KEYWORDS: Numerical Method, Boundary Element Method, Laplace Equation, Boundary Conditions, Weighted Residual Methods, Point Collocation Method, Galerkin Method, Numerical Solution For Boundary Integral Equation.

INTRODUCTION

The Boundary Element Method is defined as type of a numerical method that approximates solutions of boundary value problems (BVPs). The method is moderately new methods as it came into exist in the sixties. Contrasted with the finite element method (FEM), the improvement of the Boundary Element Method has been significantly slower. One purpose behind this slower advancement in the Boundary Element Method is the constrained accessibility of basic solutions of the boundary value problems. Another reason is probably going to be the association of singular integral equations that need to be solved. In present era, these equations well-understood, and the number of application fields in which the Boundary Element Method is used is large, although not as large as for the finite element method.

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LAPLACE EQUATION AND BOUNDARY CONDITIONS

The Laplace equation takes the form

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \nabla.(\nabla\phi) = 0 ------- (1)$$

Where 'φ' is unknown field

This equation is subjected to the following boundary conditions.

DIRICHLET BOUNDARY CONDITION

Let $\Re$ be a the closed region of the plane and let $\partial\Re$ denote the boundary of $\Re$ find the function $\phi(x, y)$ such that,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad \forall\ (x, y) \in \Re$$

$$\phi(x,y) = \phi_0(x,y) \quad \forall (x,y) \in \partial\Re ----(2)$$

NEUMANN BOUNDARY CONDITIONS

Let $\Re$ be a the closed region of the plane and let $\partial\Re$ denote the boundary of $\Re$ find the function $\phi(x, y)$ such that,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad \forall\ (x, y) \in \Re$$

$$\frac{\partial\phi}{\partial n}(x,y) = Kf(x,y) \quad \forall (x,y) \in \partial\Re ------(3)$$

WEIGHTED RESIDUAL METHODS

The majority of the numerical schemes are subsets of one formulation only, and that a specific numerical method is acquired by making certain assumptions in the general formulation, and after that the assessment of a numerical method ends up straightforward. Such a formulation which binds together all the numerical schemes is known as the weighted residual method. Although, traditionally, none of the numerical methods are derived from the weighted residual formulation, it is another way of looking at the numerical methods. The thought here is to demonstrate an elective perspective.

In the weighted residual approach, assumptions of particular functions for the unknown variables lead to different numerical schemes. The generality and the strength of each method can then be seen and compared on the same basis. The basis; of the weighted residual approach is presented next. It is shown how some of the numerical methods are obtained from the weighted residual formulation.

For example a linear differential operator 'D' is acting on a function u(x) to produce a function p(x).

$$D(u(x)) = p(x) --- (4)$$

For approximating 'u(x)' by functions $\bar{u}$ which is a linear combination of basic functions chosen from a linearly independent set. That is,

$$u(x) = \bar{u}(x) = \sum_{i=1}^{n} a_i \varphi_i$$

Now, when substituted into the differential operator, D, the result of the operations is not zero, in general. Hence an error or residual will exist:

$$E(x) = R(x) = (D(\bar{u}(x)) - p(x)) \neq 0$$

The notion in the method is to force the residual to zero in some average sense over the domain. That is

$$\int_x R(x) w_i dx = 0 \qquad i = 1,2,3......n --- (5)$$

Where the number of weight functions w_i is exactly equal the number of unknown constants a_i in $\bar{u}$. The result is a set of n algebraic equations for the unknown constants a_i.There are (at least) five weighted residual sub-methods, according to the choices for the w_i. Mainly two sub-methods of weighted residual method, point collocation and galerkin methods are explained in below sections.

POINT COLLOCATION METHOD

In this method, the weighting functions are taken from the family of Dirac δ functions in the domain. That is $w_i(x) = \delta(x - x_i)$. The Dirac δ function has the property that

$$\delta(x - x_i) = \begin{cases} 1 & \mathrm{x = x_i} \\ 0 & \text{otherwise} \end{cases}$$

Hence the integration of the weighted residual function results in the forcing of the residual to zero at specific points in the domain. That is, integration of eq. (5) with $w_i(x) = \delta(x - x_i)$ results in

$$R(x_i) = 0 - - - (6)$$

GALERKIN METHOD

This method uses the derivative of the approximating function $\bar{u}$ with respect to the unknown a_i. That is, if the function is approximated as

$$u(x) = \bar{u}(x) = \sum_{i=1}^{n} a_i \varphi_i,$$

Then the weight functions are:

$$w_i = \frac{\partial \bar{u}}{\partial a_i}$$

And the basis function is:

$$w_i = \frac{\partial \bar{u}}{\partial a_i} = \varphi_i(x) - - - (7)$$

GREEN'S IDENTITIES

The three Green's identities are three vector derivative/integral identities which can be derived starting with the vector derivative identities and their representation are shown in below:

Green's first identity

$$\int_{\Omega} \nabla\phi . \nabla w \, d\Omega + \int_{\Omega} \phi \nabla^2 w \, d\Omega = \int_{\partial s} \phi \nabla w . n \, ds - - - (8)$$

Green's second identity

$$\int_{\Omega} (\phi \nabla^2 w - w \nabla^2 \phi) \, d\Omega = \int_{\partial s} (\phi \nabla w - w \nabla \phi) . n \, ds - - - (9)$$

DERIVATION OF BOUNDARY ELEMENT METHOD IN 2D

Consider Laplace equation and multiply that equation with w and integrate over domain

$$\nabla^2 \phi = 0 \Rightarrow w[\nabla^2 \phi = 0] \Rightarrow \int_{\Omega} \mathrm{w} \nabla^2 \phi \, d\Omega = 0 - - - (10)$$

Expanding the above equation and performing the integration by parts for eq. (10)

$$\int_{\Omega} \mathrm{w} \, \frac{\partial^2 \phi}{\partial x^2} \, d\Omega + \int_{\Omega} w \, \frac{\partial^2 \phi}{\partial y^2} \, d\Omega = 0$$

$$\mathrm{w.} \, \frac{\partial \phi}{\partial \mathrm{x}} - \int_{\Omega} \frac{\partial \phi}{\partial x} . \frac{\partial w}{\partial x} \, d\Omega + \mathrm{w.} \, \frac{\partial \phi}{\partial \mathrm{y}} - \int_{\Omega} \frac{\partial \phi}{\partial y} . \frac{\partial w}{\partial y} \, d\Omega = 0$$

$$\int_{\Omega} \frac{\partial \phi}{\partial x} . \frac{\partial w}{\partial x} \, d\Omega + \int_{\Omega} \frac{\partial \phi}{\partial y} . \frac{\partial w}{\partial y} \, d\Omega = \left\{ w . \frac{\partial \phi}{\partial x} + w . \frac{\partial \phi}{\partial y} \right\} - - - eq\ (11)$$

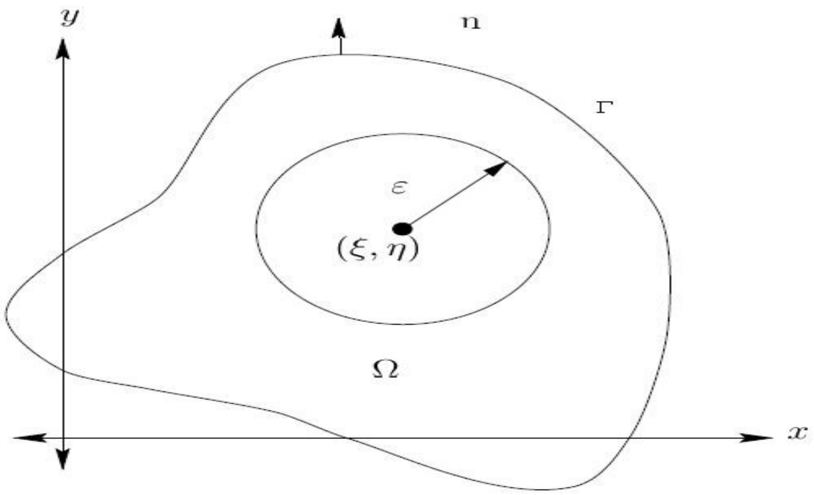

Figure 1.Generic domain for developing boundary integral equation

And again perform the integration by parts for eq. (11) for getting the second order derivative for the weighting function

$$\frac{\partial \mathrm{w}}{\partial \mathrm{x}}.\phi - \int_{\Omega} \phi \frac{\partial^2 w}{\partial x^2} d\Omega + \frac{\partial \mathrm{w}}{\partial \mathrm{y}}.\phi - \int_{\Omega} \phi \frac{\partial^2 w}{\partial y^2} d\Omega = \left\{ w(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}) \right\}$$

And following steps are the detailed steps for getting the second order derivative for the weighting function:

$$\int_{\Omega} \phi \frac{\partial^2 w}{\partial x^2} d\Omega + \int_{\Omega} \phi \frac{\partial^2 w}{\partial y^2} d\Omega = -\left\{ w(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}) \right\} + \left\{ \phi(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}) \right\}$$

Collecting all terms one side and equating to zero

$$\int_{\Omega} \phi \frac{\partial^2 w}{\partial x^2} d\Omega + \int_{\Omega} \phi \frac{\partial^2 w}{\partial y^2} d\Omega + \left\{ w(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}) \right\} - \left\{ \phi(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}) \right\} = 0$$

Finally second order derivative for the weighting function

$$\int_{\Omega} \phi(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}) d\Omega + \left\{ w(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}) \right\} - \left\{ \phi(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}) \right\} = 0$$

$$\int_{\Omega} \phi(\nabla^2 w) d\Omega + \left\{ w(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}) \right\} - \left\{ \phi(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}) \right\} = 0 --- eq(12)$$

Now from the eq.(10) and eq.(12),we can compare the R.H.S, so that

$$\int_{\Omega} \phi(\nabla^2 W) d\Omega + \left\{ W(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}) \right\} - \left\{ \phi(\frac{\partial W}{\partial x} + \frac{\partial W}{\partial y}) \right\} = \int_{\Omega} (\nabla^2 \phi) W d\Omega$$

Gathering same terms, and the equation can be modified as eq. (13)

$$\int_{\Omega} \phi(\nabla^2 w) d\Omega - \int_{\Omega} (\nabla^2 \phi) w d\Omega = \left\{ \phi(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}) \right\} - \left\{ w(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}) \right\}$$

$$\int_{\Omega} (w(\nabla^2 \phi) - \phi(\nabla^2 w)) d\Omega = \left\{ w(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}) \right\} - \left\{ \phi(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}) \right\} --- eq(13)$$

From the Green's second Identity eq. (9), the eq.(13) can be modified as

$$\int_{\Omega} (w(\nabla^2 \phi) - \phi(\nabla^2 w)) d\Omega = \int_{\Gamma} \frac{\partial \phi}{\partial n} w d\Gamma - \int_{\Gamma} \frac{\partial w}{\partial n} \phi d\Gamma --- eq(14)$$

The eq. (14) shows that Laplace operator is self-ad joint operator. In FEM simple piecewise polynomials are used as our weighting (test) functions. In BEM the fundamental solution are used so that the last term in L.H.S becomes,

$$\int_{\Omega} \phi \nabla^2 w \, d\Omega = -\int_{\Omega} \phi \delta(\varsigma - x, \eta - y) \, d\Omega = -\phi(\varsigma, \eta) --- eq(15)$$

Assuming $(\varsigma, \eta) \in \Omega$, not on the boundary ($\Gamma$) and $\int_{\Omega} w(\nabla^2 \phi) d\Omega = 0$. Now substituting eq. (15) in eq. (14), then

The Boundary integral equation

$$-\phi(\varsigma, \eta) = \int_{\Gamma} \frac{\partial \phi}{\partial n} w d\Gamma - \int_{\Gamma} \frac{\partial w}{\partial n} \phi d\Gamma --- eq.(16)$$

For $(\varsigma, \eta) \in \Omega$.

The above eq. (22) explains that we can (in theory) find u at an arbitrary point P $(\varsigma, \eta) \in \Omega$ by looking at ϕ and w only on the boundary. This still doesn't help us unless we know ϕ and/or $\partial \phi / \partial n$ on the boundary.

For representing the boundary (Γ) behavior for given domain (Ω), the constant function

'c (P)' is included in above equation

$$-c(P)\phi(P) = \int_{\Gamma} \frac{\partial \phi}{\partial n} w d\Gamma - \int_{\Gamma} \frac{\partial w}{\partial n} \phi d\Gamma --- eq.(17)$$

Where P is arbitrary point of the domain and the following equation gives the definition of c(P) function.

$$c(p) = \begin{cases} 1 & P \in \Omega \\ \frac{1}{2} & P \in \Gamma \quad \Gamma \text{ is smooth} \\ (1 - \frac{\alpha}{2\Pi}) & P \in \Gamma \quad \Gamma \text{ is not smooth} \\ 0 & P \in \Omega \end{cases}$$

NUMERICAL SOLUTION FOR BOUNDARY INTEGRAL EQUATION

The initial step for solving Boundary Integral equation we need to discretize the boundary surface (Γ) of the given domain (Ω) into finite set of boundary elements:

$$\Gamma = \bigcup_{j=1}^{N} \Gamma_j --- (18)$$

Now the eq. (23) can be modified as the following equation

$$c(P)\phi(P)+\sum_{j=1}^{N}\int_{\Gamma_j}\frac{\partial w}{\partial n}\phi d\Gamma=\sum_{j=1}^{N}\int_{\Gamma_j}\frac{\partial\phi}{\partial n}wd\Gamma----eq.(19)$$

For each boundary element Γ_j we introduce standard basis functions

$$\phi_j=\sum_\alpha\varphi_\alpha\phi_{j\alpha}\quad and\quad q_j=\frac{\partial\phi_j}{\partial n}=\sum_\alpha\varphi_\alpha q_{j\alpha}$$

Where ϕ_j, q_j are values of ϕ, q on element Γ_j and $\phi_{j\alpha}, q_{j\alpha}$ are values of ϕ, q and q at node α on Element Γ_j

This basis functions for ϕ and q can be any of the standard one-dimensional finite element basis functions. In general the basic functions used for ϕ u and q do not have to be the same and these basis functions can even be different to the basic functions used for the geometry, but are generally taken to be the same.

From above considerations

$$c(P)\phi(P)+\sum_{j=1}^{N}\sum_\alpha\phi_{j\alpha}\int_{\Gamma_j}\varphi_\alpha\frac{\partial w}{\partial n}\phi d\Gamma=\sum_{j=1}^{N}\sum_\alpha q_{j\alpha}\int_{\Gamma_j}\varphi_\alpha wd\Gamma----eq.(20)$$

This equation holds for any point PP on the surface (Γ). We now generate one equation per node by putting the point P to be at each node in turn. If P is at node 'i' say, then we have

$$c_i\phi_i+\sum_{j=1}^{N}\sum_\alpha\phi_{j\alpha}\int_{\Gamma_j}\varphi_\alpha\frac{\partial w_i}{\partial n}\phi d\Gamma=\sum_{j=1}^{N}\sum_\alpha q_{j\alpha}\int_{\Gamma_j}\varphi_\alpha w_i d\Gamma----eq.(21)$$

Where w_i is the fundamental solution with the singularity at node 'i'. Now we can rewrite Equation (21) in a more abbreviated form as

$$c_i\phi_i+\sum_{j=1}^{N}\sum_\alpha\phi_{j\alpha}a_{ij}^\alpha=\sum_{j=1}^{N}\sum_\alpha q_{j\alpha}b_{ij}^\alpha---eq.(22)$$

Where $a_{ij}^\alpha, b_{ij}^\alpha$ are

$$a_{ij}^\alpha=\int_{\Gamma_j}\phi_\alpha\frac{\partial W_i}{\partial n}d\Gamma,\ b_{ij}^\alpha=\int_{\Gamma_j}\phi_\alpha W_i d\Gamma$$

The equation eq. (22) is for node 'i' and if we have 'L' L nodes, then we can generate 'L' equations.

We can gather these equations into the matrix system

$$A\phi=Bq---eq.(23)$$

ϕ ϕ ϕ

Where the vectors and q are the vectors of nodal values of and q. at each node, we must specify either a value of or q (or some combination of these) to have a well-defined problem. We therefore have L equations (the number of nodes) and have L unknowns to find. We need to rearrange the above system of equations to get

$$CX=F---eq.(24)$$

Where 'X' is an unknown vector. This can be solved using standard linear equation solvers, although specialist solvers are required if the problem is large.

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