International Journal of Transformation in Applied Mathematics & Statistics
Vol. 3, Issue 1 – 2020
ISSN: 2581-7620

Adomian Decomposition Method and its Applications for Fractional Differential Equations

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Abstract

This topic is devoted to the decomposition of fractional differential equations with the help of Adomian polynomials. This method has wide range of applications. Here we will discuss a few applications. This method is easy to programme and implement in initial value problems. Partial differential equation with non-liner coefficient play a very important role. This method can be used without the need of discretization or perturbation.

Keywords: Adomian Composition, Variable Coefficients, Fractional differential equations.

Introduction

The Adomian Decomposition method was given by Adomian. In method we used to split our given differential equation into linear and non linear compositions and then inverting the higher order derivative operator contained in the linear operator on both sides. After that recognising the initial conditions and the first term of the series solution. Then we decompose non linear function in terms of a polynomial, specially named after this method, the Adomian polynomial and then we find the successive terms of the series by using the recursion terms of the polynomials. This method has been successfully tested and verified for large class of linear algebraic equations, differential equations, partial differential equations, system of differential as well as partial differential equations. We will talk about the solution of non-homogeneous fractional differential equations (FDE). Also using ADM we will simplify a few problems of NHFDE in the subsequent part of the paper.

Fractional Differential equation

A Fractional differential equation of the form.

$\left(D^{\alpha_n}+a_{n-1}D^{\alpha_{n-1}}+\ldots+a_1D^{\alpha_1}+Q_0\right)y(t)=f(t)$ with the cond^n $y^{(k)}$ (c) = b_k $\forall k = 0, 1, 2.... n - 1$ is known as a linear fractional differential equation which is non-homogeneous.

Adomian Decomposition Method

This method involves the formation of the series sum of the problem. We will solve the FDE using approximation over the solution and using the Adomain polynon: Adomain polynomials are considered as the approximate solution. Then we get the approximate solution of the given FDE.

Adomain Polynomials

We denote these polynomials with A_0, A_1, A_2, ... A_n,.... here from any given FDE. We consider the initials as y(to) = y_0 such that the decomposition of the unknown function y (t) can be written as a series sum of y_0, y_1, y_3 where each y_n can be written in form of the Adomain polynomials A_0, A_1, A_2, ...

The Adomain Polynomials are defined as follows:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(t, \sum_{j=0}^{n} y_j \lambda^j\right)\right]_{\lambda=0}$$

These polynomials can be calculated step by stop for n = 0, 1, 2, 3,.... with the starting value A_0 (t) = F (t =1 , y_0 (t)). Now using the value of A_0 (t) we will find y_1 (t) and then using value of y_1 we can write A_1 similarly, we can find y_2 with the help of A_1 and the using y_2, we define A_2. So repeating this process. We get finally A_0, A_1, A_2 ...., A_n .... then the solution can be constructed as y(t) = y_0 (t) + y_1 (t) + y_2 (t) + ...

Applications:

1. Solve $D^\alpha y = t^2 + y^2$ with y (0) = 0, y' (0) = 1

Solution: Apply Laplace transform

$$L\left(D^\alpha y(t)\right) = s^\alpha y - s^{\alpha-1} y(0) - s^{\alpha-2} y'(0) \quad \text{where}$$

$$= s^\alpha \; y_{(s)} - s^{\alpha-2} = s^\alpha (y-s)^{-2y} = L\left(y(t)\right)$$

R.H.S. $L\left(t^2 + y^2\right)$

$$\Rightarrow \quad s^\alpha\left(y - \frac{1}{s^2}\right) = L\left(t^2 + y^2\right)$$

$$y = \frac{1}{s^2} + \frac{1}{s^\alpha} + \frac{1}{s^\alpha} L\left(t^2 + y^2\right) \qquad (1)$$

For the decomposition

$$y(t)\sum_{n=0}^{\infty} y_n = y_0 + y_1 + y_2 + \ldots \qquad (2)$$

so we get

$$y = \sum_{n=0}^{\infty} y_n \quad \text{and} \quad t^2 + y^2 = \sum_{n=0}^{\infty} A_n \qquad (3)$$

Where A_n are adomian polynomials

$$A_0(t) = t^2 + y_0^2\ , \qquad A_1(t) = 2y_0y_1 \text{ from}$$

$$A_2 = y_1^2 + 2y_0y_2 \quad , \quad A_3 = 2y_1y_2 + 2y_0y_3 \qquad (4)$$

Now substitute equation (3) and (4) in equation (1)

$$\sum_{n=0}^{\infty} y_n = \frac{1}{s^2} + \frac{1}{s^\alpha} L\left(\sum_{n=0}^{\infty} A_n\right)$$

$$y_0 + y_1 + y_2 + \ldots\ldots = \frac{1}{s^2} + \frac{1}{s^\alpha} L\left(A_0 + A_1 + A_2 + \ldots\ldots.\right)$$

$$y_0 = \frac{1}{s^2} \Rightarrow y_0 = t \qquad \left[\because\ L^{-1}\frac{1}{s^2} = t\right]$$

$$y_1 = \frac{1}{s^\alpha} L\, A_0(t) = \frac{1}{s^\alpha} L\left(t^2 + t^2\right) = \frac{2}{s^\alpha}.\frac{2!}{s^3} = \frac{4}{s^{\alpha+3}} = \frac{4}{s^{\alpha+2+1}}$$

$$\Rightarrow \qquad y_1 = 4\,.\,\frac{t^{\alpha+2}}{(\alpha+2)!}$$

$$y_2 = \frac{1}{s^\alpha}.L(A_1) = \frac{1}{s^\alpha} L\left(2.t^2.\frac{4t^{\alpha+2}}{(\alpha+t)!}\right) = \frac{8}{(\alpha+2)!s^\alpha}.L\left[t^{\alpha+4}\right]$$

$$= \frac{8}{(\alpha+1)!}\frac{(\alpha+4)!}{s^{2\alpha+5}} = 8(\alpha+4)(\alpha+3)\frac{1}{s^{2\alpha+5}}$$

$$y_2 = 8(\alpha+4)(\alpha+3)L^{-1}\left[\frac{1}{s^{2\alpha+4+1}}\right] = 8(\alpha+4)(\alpha+3)\frac{t^{2\alpha+2}}{(2\alpha+4)!}$$

$$y_3 = \frac{1}{s^\alpha} L(A_s) = \frac{1}{s^2} L\left(y_1^2 + 2y_0 y_2\right)$$

$$= \frac{1}{s^\alpha} L\left[\left(\frac{4t^{\alpha+2}}{(\alpha+2)!}\right)^2 + 2 + 8(\alpha+4)(\alpha+3)\frac{t^{2\alpha+4}}{(2\alpha+4)!}\right]$$

$$= \frac{1}{s^\alpha} \cdot \frac{16}{\left[(\alpha+2)!\right]^2} L\left(t^{2\alpha+4}\right) + \frac{1}{s^\alpha} \frac{16.(\alpha+4)(\alpha+3)}{(2\alpha+4)!} L\left(t^{2\alpha+5}\right)$$

$$= \frac{16}{\left[(\alpha+2)!\right]^2} \frac{1}{s^\alpha} \cdot \frac{(2\alpha+4)}{s^{2\alpha+5}} + \frac{16(\alpha+4)(\alpha+3)}{\cancel{(2\alpha+4)}!} \frac{1}{s^\alpha} \frac{(2\alpha+5)!}{s^{2\alpha+6}}$$

$$= \frac{16(2\alpha+4)!}{\left[(\alpha+2)!\right]^2} \frac{1}{s^{3\alpha+5}} + 16(\alpha+4)(\alpha+3)(2\alpha+5) \times \frac{1}{s^{3\alpha+6}}$$

$$y_3 = \frac{16(2\alpha+4)!}{\left[(\alpha+2)!\right]^2} + \frac{3\alpha+4}{(3\alpha+4)!} + 16(\alpha+4)(\alpha+3)(2\alpha+5)\frac{t}{(3\alpha+5)!}$$

$$= \frac{16t^{3\alpha+4}}{(3\alpha+4)!}\left[\frac{(2\alpha+4)!}{\left[(\alpha+2)!\right]^2} + \frac{(\alpha+4)(\alpha+3)(2\alpha+5)t}{3\alpha+5}\right]$$

And finally it yields :

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots\ldots\ldots$$

$$y(t) = t + \frac{4t^{\alpha+2}}{(\alpha+2)!} + \frac{8(\alpha+4)(\alpha+3)t^{2\alpha+4}}{(2\alpha+4)!} + \frac{16t^{3\alpha+4}}{(3\alpha+4)!}\left[\frac{(2\alpha+4)}{\left[(\alpha+2)!\right]^2} + \frac{(\alpha+4)(\alpha+3)(\alpha+5)t}{3\alpha+5}\right] \ldots\ldots$$

for α = 2

$$D^2 y = t^2 + y^2$$

$$y^1(+) = t^2 + y^2 \qquad y(0) = 0, \qquad y'(0) = 1$$

for α = 2

$$y(t) = t + \frac{4.t^4}{4!} + \frac{\cancel{8} \times 30}{\cancel{8}!.7!} t^8 + \frac{16.t^{10}}{10!}\left[\frac{8!}{(4!)^2} + \frac{270}{11} t\right] + \ldots$$

$$=t+\frac{t^4}{3!}+\frac{t^8}{168}+16\frac{t^{10}}{10!}\left[70+\frac{270}{11}t\right]+\ldots$$

$$=t+\frac{t^4}{6}+\frac{t^7}{168}+\frac{t^{10}}{3240}+\frac{t^{11}}{9240}\ldots$$

2. Solve FDE

$$D^{2x}y(t)=e^k y(t) \quad 0<\alpha\le 1$$

$$1=y(0)\,y^{\alpha}(0)=0$$

Solution: Apply Laplace transform

$$L\left[D^{2\alpha}y(t)\right]=L\left(e^{ky(t)}\right)$$

if $L\left[y(t)\right]=y$ then

$$L\left[D^{2\alpha}y(t)\right]=s^{2\alpha}y-s^{2\alpha-1}y(0)$$

$$=s^{2\alpha}y-s^{2\alpha-1}$$

$$\Rightarrow \quad s^{2\alpha}\left[y-\frac{1}{s}\right]=L\left[e^{ky(t)}\right]$$

$$y=\frac{1}{s}+\frac{1}{s^{2\alpha}}L\left[e^{ky(t)}\right] \tag{1}$$

Now to decompose, we put

$$y=\sum_{n=0}^{\infty}y_n(t) \tag{2}$$

we get

$$y(t)=\sum_{n=0}^{\infty}y_n(t) \quad , \quad e^{ky(t)}=\sum_{n=0}^{\infty}A_n \tag{3}$$

using equation (2) and (3) in equation (1)

$$\sum_{n=0}^{\infty}y_n=\frac{1}{s}+\frac{1}{s^{2\alpha}}.\,L\left[\sum_{n=0}^{\infty}A_n\right]$$

$$= \frac{1}{s} + \frac{1}{s^{2\alpha}}.L\left[A_0 + A_1 + A_2 + ...\right]$$

where

$$A_0 + A_1 + A_2.... = e^{ky(t)} = e^{k\left[y_0 + y_1 + y_2 + ...\right]}$$

$$= e^{ky}\left[e^{k\left[y_1 + y_2 + ...\right]}\right]$$

$$A_0 + A_1 + A_2 + ... = e^{ky_0}\left[1 + K\left(y_1 + y_2 + ...y_n + ...\right) + k^2\frac{\left(y_1 + y_2 + ...\right)^2}{2!} + ...\right]$$

$$= e^{ky_0} + k^{eky_0}\left[y_1 + y_2 + ...\right] + k^2 e^{ky_0}\left[y_1^2 + y_2^+ + 2y_1y_2 + 2y_1y_3 + 2y_2y_3\right]_{..}$$

$$\Rightarrow \qquad A_0 = e^{ky_0},\ A_1 = ky_1\ e^{ky_0}$$

$$A_2 = e^{ky_0}\left[ky_2 + k^2y_1^2\right],\ A_3 = ky_3e^{ky_0} + 2k^2y_1y_2e^{ky_0} + k^3y_1^3e^{ky}$$

Then

$$y_0 \quad = \quad \frac{1}{s} \Rightarrow y_0 = 1$$

$$y_1 = \frac{L\left[e^{ky_0}\right]}{s^{2\alpha}} = \frac{L\left[e^k\right]}{s^{2\alpha}} = e^k.\frac{1}{s^{2\alpha+1}}$$

$$\Rightarrow \qquad y_1 = e^k\frac{t^{2\alpha}}{(2\alpha)!}$$

$$y_2 = \frac{1}{s^{2\alpha}}L\left[A_1\right] = \frac{1}{s^{2\alpha}}L\left[ky_1e^{ky_0}\right]$$

$$= \frac{1}{s^{2\alpha}}L\left(k.e^k\frac{t^{2\alpha}}{(2d)!}.e^k\right)$$

$$= \frac{1}{s^{2\alpha}}\frac{k.e^{2k}}{(2\alpha)!}L\left[t^{2\alpha}\right]$$

$$= \frac{ke^{2k}}{(2\alpha)!}\frac{(2\alpha)!}{s^{4\alpha+1}} = ke^{2k}.\frac{1}{s^{4\alpha+1}}$$

$$y_2 = k.e^{2k}.\frac{t^{4\alpha}}{(4\alpha)!}$$

$$y_3 = \frac{1}{s^{2\alpha}}.L.(A_2) = \frac{1}{s^{2\alpha}}.L\left[e^{k.}\left(k.ke^{2k}\frac{t^{4\alpha}}{(4\alpha)} + k^2e^{2k}\frac{t^{4\alpha}}{((2\alpha)1)^2}\right)\right]$$

$$= \frac{e^k k^2 e^{2k}}{s^{2\alpha}}L\left[\frac{t}{(4\alpha)!} + \frac{t}{[(2\alpha)!]^2}\right]$$

$$y_3 = e^{3k}.k^2\left(\frac{1}{(4\alpha)!} + \frac{1}{[(2\alpha)!]^2}\right).\frac{1}{s^{2\alpha}}.\frac{(4\alpha)!}{s^{4\alpha+1}}$$

$$y_3 = k^2e^k\left(\frac{1}{(4\alpha)!} + \frac{1}{[(2\alpha)]^2}\right)(4\alpha)!\,\frac{t^{6\alpha}}{(6\alpha)!}$$

So the solution can be written as :

$$y(t) = 1 + e^k\frac{t^{2\alpha}}{(2\alpha)!} + ke^{2k}\frac{t^{4\alpha}}{(4\alpha)!} + k^2e^{3k}\left(\frac{1}{(4\alpha)!} + \frac{1}{[(2\alpha)!]^2}\right).\frac{(4\alpha)!.t^{6\alpha}}{(6\alpha)!}$$ sd

Now for α = 1

$$y(t) = 1 + e^k\frac{t^2}{2!} + ke^{2k}\frac{t^4}{4!} + k^2e^{3k}\left(\frac{1}{4!} + \frac{1}{4}\right).\frac{4!.t^6}{6!} + \ldots$$

$$= 1 + \frac{e^k}{2}t^2 + \frac{ke^{2k}}{24}t^4 + \frac{7k^3e^{3k}}{720}t^6 + \ldots\ldots$$

3. Solve the FDE

$$D^\alpha y(t) = y(t) \qquad 0 < \alpha \le 1$$
$$y(0) = 1$$

Solution: Here apply the Laplace transform

$$L[D^\alpha y(t)] = L[y(t)]$$

$$s^\alpha y(t) = s^{\alpha-1}y(0) \quad = \quad L[y(t)] \text{ where } y = [L(y+t)]$$

$$\Rightarrow \quad S^{\alpha}\left[y-\frac{1}{s}\right]=L\left(y(t)\right)$$

$$y=\frac{1}{s}+\frac{1}{s^{\alpha}}L\left(y(t)\right) \tag{1}$$

For decomposition, we take

$$y=\sum_{n=0}^{\infty}y_n \tag{2}$$

we get

$$y=\sum_{n=0}^{\infty}y_n(t) \quad , \qquad y(t)=\sum_{n=0}^{\infty}A_n \tag{3}$$

Comparing the above two equation we find $A_n = y_n \quad \forall n$

Now

$$y_0(t)=\frac{1}{s} \Rightarrow y_0(t)=1$$

then $A_0(t)=y_0(t)=1$

Now

$$y_1(t)=\frac{1}{s^{\alpha}}.\,L[A_0]=\frac{1}{s^{\alpha}}.\,L[1]=\frac{1}{s^{\alpha}}.\frac{1}{s}$$

$$\Rightarrow \quad y(t)=\frac{t^{\alpha}}{\alpha!}\ then\ A_1(t)=y_1(t)=\frac{t^{\alpha}}{\alpha!}$$

$$y_2=\frac{1}{s^{\alpha}}L[A_1]=\frac{1}{s^{\alpha}}L\left[\frac{t^{\alpha}}{\alpha!}\right]=\frac{1}{\cancel{\alpha!}}\frac{1}{s^{\alpha}}.\frac{\cancel{\alpha!}}{s^{\alpha+1}}=\frac{1}{s^{2\alpha+1}}$$

$$\Rightarrow \quad y_2=\frac{t^{2}}{(2\alpha)!}$$

$$\Rightarrow \quad A_2=y_2=\frac{t^{2\alpha}}{(2\alpha)!}$$

Now

$$y_3 = \frac{1}{s^\alpha} L[A_2] = \frac{1}{s^2}.L\left[\frac{t^{2\alpha}}{(2\alpha)!}\right] = \frac{1}{(2\alpha)!}\,\frac{1}{s^\alpha}$$

$$y_3 = \frac{1}{s^{3\alpha+1}} \quad \Rightarrow \quad y_3 = \frac{t^{3\alpha}}{(3\alpha)!}$$

So the solution can be written as

$$y(t) = y_0 + y_1 + y_2 + y_3 + \ldots\ldots$$

$$= 1 + \frac{t^\alpha}{\alpha!} + \frac{t^{2\alpha}}{(2\alpha)!} + \frac{t^{3\alpha}}{(3\alpha)!} + \ldots\ldots$$

Now for α = 1 given F.D.E is y' = y y (0) =1 then the above solution will be

$$y(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\ldots.$$

$$= e^t$$

This method gives a very high accuracy in results

Conclusion

In this paper, we showed the applicability and simplicity of AD M applied to Non-hom FDE. This method is very helpful and efficient technique for solving different types of problems. This method solves the equation without any undergoing to process of discretization, perturbation or linearlization.

The method was applied on three different examples where the numerical solutions clearly shows that the ADM provides very accurate results which are very close to the exact solution.

Thus, the method is capable of giving quick solutions with high accuracy and reliability in solving these equations.

References

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