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Arithmetical Functions and Fibonacci Numbers

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Abstract

The first large collection of tables of indices for all primes less than 1,000 was the "Canon Arithmeticus", computed under the direction of Jacobi. It was published in 1839. A revised and extended version was re-published in 1956. Since then, high-speed computing machines have been used to extend the range of these tables. This paper explains the arithmetical functions and presents a brief discussion about Fibonacci numbers in a suitable form.

1. Introduction

Functions f(n) of the positive integer n, which express some arithmetical property of n are called arithmetical functions. ∅(n) is the number of positive integers less than n that are relatively prime to n, with the value ∅(1) =1 defined separately. Here, we represent

the number of positive divisors of n by d(n),

the sum of the positive divisors of n by σ(n), &

the sum of the k^{th} powers of the positive divisors of n by $\sigma_{k(n)}$

The divisor-function, $\sigma_{k(n)}$ can be expressed as-

$$\sigma_{k(n)} = \sum_{\frac{d}{n}} d^k \tag{1.1}$$

where the sum is extended over all positive divisors of n. The functions d(n) and σ(n) are special cases of $\sigma_{k(n)}$ with

$\left.\begin{matrix} k=0 \\ \& \ k=1 \end{matrix}\right\}$ respectively,

These illustrations have the two special properties:-

$$\left.\begin{matrix} f(1)=1 \\ \& \ f(mn)=f(m)f(n) \end{matrix}\right\} \tag{1.2}$$

whenever, (m, n) = 1 (1.3)

Arithmetical functions with properties (1.2) and (1.3) are called "multiplicative arithmetical functions". These functions are completely determined by their values at the prime power. If n is a product of powers of distinct primes, call $n=p^a q^b \ldots$, then

$f(n)= f(p^a)f(q^b) \ldots.$ (1.4)

Illustration can be put as-

$\sigma_{2(24)} = \sigma_{2(3)} \cdot \sigma_{(2^3)}$

$= (1^2+3^2)(1^2+2^2+4^2+8^2)$

$= 10\times85 = 850$

The values of foregoing functions at the prime powers are given by

$\emptyset(p^a) = p^a - p^{a-1}$

$d(p^a) = a+1$

$\sigma_{(p^a)} = 1+p+p^2 + \ldots + p^a = \frac{p^{a+1}-1}{p-1}$

$\sigma_{k(p^a)} = \frac{p^{ak+k}-1}{p^k-1}$, if $k\neq0$ (1.5)

Any arithmetical function f(n) can be used to generate another,

$f(n) = \sum_{\frac{d}{n}} f(d)$ (1.6)

by summing f(d) over all positive divisors d of n.

Equation (1.6) can also be written as-

$F(n) = \sum_{\frac{d}{n}} f(\frac{n}{d})$ (1.7)

If the function f(n) is multiplicative, so is the divisor sum F(n), and conversely, if the divisor sum F(n) is multiplicative, so is f(n).

2. Arithmetical Function and Mobius Function

The German mathematician, August Ferdinand Mobius suggested that any arithmetical function f(n) can be determined from a knowledge of its divisor-sum F(n), commonly known as "Mobius inversion formula",

$f(n) = \sum_{\frac{d}{n}} \mu\left(\frac{d}{n}\right) F(\frac{n}{d})$ (2.1)

where $\mu(n)$ is a multiplicative arithmetical function, called the Mobius function. Its values are 0, 1 and -1 & are obtained as follows:

$\mu(1) = 1,$

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$\mu(p) = -1$ for each prime p, &

$\mu(p^k) = 0$, if $k>1$.

Because $\mu(n)$ is multiplicative, $\mu(n) = 0$, if n is divisible by the square of any prime, and $\mu(n) = (-1)^r$ if n is the product of r- distinct primes. The divisor sum $\sum_{\frac{d}{n}} \mu(d)$ is equal to zero, if $n>1$ and $\sum_{\frac{d}{n}} \mu(d) = 1$, if $n=1$.

The Mobius inversion formula often reveals hidden properties of arithmetical functions. It can be used to prove that the Euler function $\emptyset(n)$ is multiplicative, a result, that is not immediately evident. Every integer k for 1 to n has a greatest common divisor (GCD) d = (k, n) with n; then $\frac{k}{d}$ and $\frac{n}{d}$ are relatively prime. Just as $\emptyset(n)$ counts the number of integers from 1 to n that are relatively prime to n, so $\emptyset(\frac{n}{d})$ counts the number of integers $q=\frac{k}{d}$ from 1 to $\frac{n}{d}$ that are relatively prime to $\frac{n}{d}$. So, $\emptyset(\frac{n}{d})$ is also the number of k, for which

(k, n) = d. But there are n- values of k, altogether, so we have,

$$n = \sum_{\frac{d}{n}} \emptyset\left(\frac{d}{n}\right)$$

That is , $n = \sum_{\frac{d}{n}} f(d)$ (2.2)

The Mobius inversion- formula gives

$$\emptyset(n) = \sum_{\frac{d}{n}} \mu(d) \frac{n}{d} \quad (2.3)$$

Alternatively, we can also write equation (2.3) as-

$$\frac{\emptyset(n)}{n} = \sum_{\frac{d}{n}} \frac{\mu(d)}{d} \quad (2.4)$$

Equation (2.4) shows that $\frac{\emptyset(n)}{n}$ is the divisor sum of the multiplicative function $\frac{\mu(n)}{n}$ so $\frac{\emptyset(n)}{n}$ is also multiplicative, hence $\emptyset(n)$ is multiplicative.

3. Arithmetical Function and Fibonacci Numbers

Some arithmetical functions are described by giving the first two function- values f(1) and then expressing f(n) for $n \geq 3$ in terms of f(n-1) and f(n-2). Mathematically, Fibonacci-numbers are defined by

$$f(n) = f(n-1)+f(n-2) \quad (3.1)$$

$$\text{provided} \left.\begin{matrix} f(1)=1 \\ \& f(2)=1 \end{matrix}\right\} \quad (3.2)$$

The functional values of f(n) are denoted by F_n. The first few terms are

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377.

Fibonacci n umbers have many interesting properties, for example,

$$F_{2n+1} = 1+F_2 +F_4+ F_6+ . . . . + F_{2n} \quad (3.3)$$

Regarding the matrix-multiplication, it is evident to say that the n-th of the matrix,

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}_{2\times2}$ is

$$A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}_{2\times2} \quad (3.4)$$

for every n≥2.

The relation,

$$(-1)^n = F_{n+1}F_{n-1} - F_n^2 \quad (3.5)$$

is obtained by equating determinants in the matrix equation, (3.4). Fibonacci numbers have also an interesting divisibility property.

$\frac{F_n}{F_m}$ whenever $\frac{n}{m}$.

For example,

$\frac{F_7}{F_{14}}$ because $\frac{7}{14}$. This can be verified because F_7 =13 & F_{14}= 377 =13.29.

Conclusions: Fibonacci numbers illustrate a linear recurrence sequence, which is given by any two starting numbers x_1 and x_2 coupled with a recurrence relation of the form:

$x_n = Ax_{n-1} + Bx_{n-2}$, for n≥3,

where A & B are given constants. This relation gives the process of determination of x_n from the two earlier terms x_{n-1} and x_{n-2} . Explicit formulae are available for expressing x_n in terms of x_1 , x_2, and the roots of the quadratic equation

$x^2 = Ax + B$.

The formula for the Fibonacci numbers is –

$F_n = \frac{\alpha^n+\beta^n}{\alpha-\beta}$, where α and β are the roots of the quadratic equation, $x^2 = x+1$ given by-

$\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$

The number α, which also appears as a ratio in various geometrics figures, is called the golden ratio. The ancient Greeks considered rectangles whose sides have ratio α to be the most pleasantly proportioned of all rectangles, aside from the square. They called them golden rectangles and believed that, for the ideal beauty of any figure including the human-form, proportions of the golden ratio.

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