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Postulate of Collinearity and Parallelism

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Abstract

In this article, we would frame a new postulate, postulate of collinearity and parallelism, in order to prove parallel postulate as a theorem; hence getting a solution to the chaos regarding Euclid's fifth postulate. We would also prove some other postulates (axioms) to be theorems and indeed prove the derived propositions of parallel postulate in order to conclude the postulate of collinearity and parallelism worthy to be a substitute of the fifth postulate of Euclid.

Keywords: Postulate of collinearity and parallelism, Parallel postulate, Triangle postulate, Euclidean geometry, Playfair's axiom.

Introduction

Euclidean geometry was formulated around 300 B.C.E. by Euclid of Alexandria. In his text *Elements*, Euclid proposed five postulates. The fifth postulate, better known as the parallel postulate, has been a matter of heated debates between mathematicians for centuries. According to several mathematicians across the globe, the parallel postulate doesn't seem to fit into the group of postulates; but should have been a theorem instead. There were many attempts to prove the considered postulate to be a theorem; unfortunately, all attempts to do so were doomed to failure.

Non-Euclidean geometry is undoubtedly a solution to the problem, as in practical situations, we deal with spherical and other similar surfaces like the Earth. But the base of practical problems which deal with the situations in an Euclidean plane, rely on Euclidean geometry. So an extended idea and correction in this branch, would rise a new conception and a distinct section of geometry.

This introduces the necessity for another postulate which would be a simpler statement and worthy to replace the parallel postulate. Hence framing the postulate of collinearity and parallelism.

Initially, we would proceed to describe the mentioned postulate in Section 2, followed by proving the parallel postulate a theorem by using the first four postulates of Euclid and the postulate of collinearity and parallelism in Section 3. We will also prove the derivations and propositions derived from the parallel postulate using the same in Section 4.

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Postulate of collinearity and parallelism

A. Introduction to postulate of collinearity and parallelism

The postulate of collinearity and parallelism can be stated as:

If two or more points are equidistant from a line, then they are collinear and the line joining those, is parallel to the initial line.

Clearly, the statement above is more worthy to be a postulate than the then stated parallel postulate. We may now proceed to explain the postulate of collinearity and parallelism in details.

B. Description

Let us assume a line l (as in Figure 1). There are points $n_1, n_2, n_3, \ldots, n_k$ not on l and $k = \{x \mid x \in \mathbb{N} \text{ and } 2 \leq x\}$. Suppose $n_1A_1, n_2A_2, \ldots, n_kA_k$ are perpendiculars to l and $n_1A_1 = n_2A_2 = n_3A_3 = \ldots = n_kA_k = h$. Since the distances of $n_1, n_2, n_3, \ldots, n_k$ from l are equal, then the postulate of collinearity and parallelism states that the points $n_1, n_2, n_3, \ldots, n_k$ are collinear and the line joining those, say m, is parallel to l.

Notably, if $h = 0$ i.e. if the points $n_1, n_2, n_3, \ldots, n_k$ are on the line l; then m and l will coincide, without changing the conclusion of postulate of collinearity and parallelism.

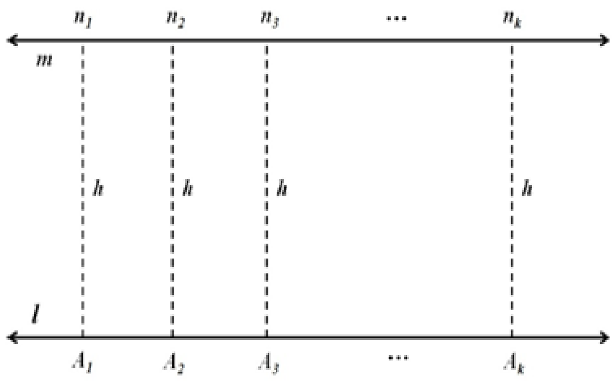

Figure 1.Postulate of collinearity and parallelism

Proving parallel postulate as a theorem

A. Parallel postulate

If a straight line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

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B. Description

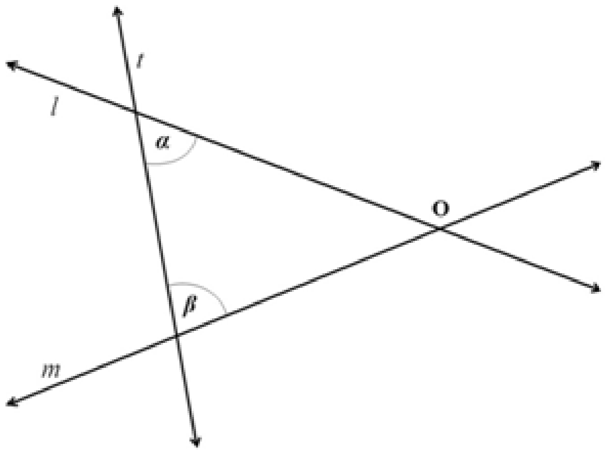

Figure 2.Parallel postulate

Let us assume a system of two lines *l* and *m* (as in Figure 2). Let *t* be the line cutting these two lines. Then there will be two angles α and β. The parallel postulate says that if (α + β) < 180˚ then *l* and *m* will meet at a point, say O. By considering the diagram, we may easily predict that the postulate can be proved by the triangle postulate which says that the sum of interior angles of all triangles equal 180˚. But it's worth mentioning that the triangle postulate is itself a derivation of the parallel postulate. So, we have to initially prove the triangle postulate independent of the parallel postulate. Hence, we use the conclusion of postulate of collinearity and parallelism in order to do so.

C. Proving triangle postulate independent of parallel postulate

To prove. The sum of all interior angles of a triangle equal

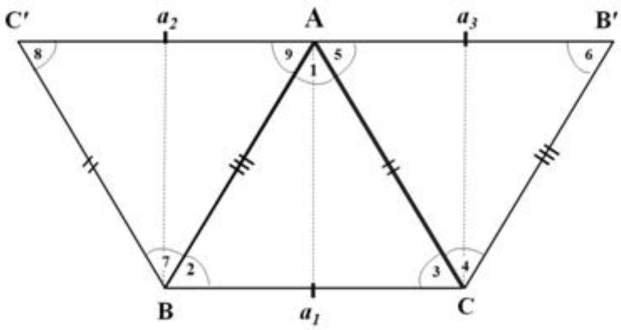

Figure 3.Proving triangle postulate

Proof. Assume any triangle ABC (as in Figure 3). Taking AB as radius and C as centre, draw an arc. Then taking BC as radius and A as centre, draw an arc to intersect the previous one at B′. We get a triangle AB′C such that ΔAB′C ≅ ΔCBA [By SSS criterion].

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Similarly, taking AC as radius and B as centre, draw an arc. Again, taking BC as radius and A as centre, draw another arc. Let the two arcs intersect at C′. We get a triangle ABC′ such that ΔABC′ ≅ ΔBAC [By SSS criterion].

∴ ΔABC ≅ ΔCB′A ≅ ΔBAC′

Since all the triangles are congruent, therefore the respective altitudes are also congruent. Let Aa_1, Ca_2 and Ba_3 be the altitudes of triangles ABC, AB′C and ABC′ respectively. Clearly, Aa_1 = Ca_2 = Ba_3. By postulate of collinearity and parallelism; a_3, A and a_2 are collinear since the points are equidistant from BC. Also ∠1 = ∠4 = ∠7; ∠3 = ∠5 = ∠8; ∠2 = ∠6 = ∠9 for the triangles are congruent. Hence, ∠1 + ∠9 + ∠5 = ∠1 + ∠2 + ∠3 = 180˚ [∵ the angles are on a line].

So, we proved that the sum of interior angles of a triangle equals 180˚.

Indeed, we didn't use the concept of parallelism and hence could also prove that the triangle postulate is independent of the parallel postulate. Now, we may use the result in order to prove the parallel postulate to be a theorem.

D. Proving parallel postulate

Proof. Let there be a system of two lines *l* and *m*, and *t* be a line cutting the system of these lines. Also, α and β be the angles formed. If the lines meet at a point, then there will be an integral and positive value of θ. So, we assume cases regarding the values of α and β to see in which cases we get the considered value of θ.

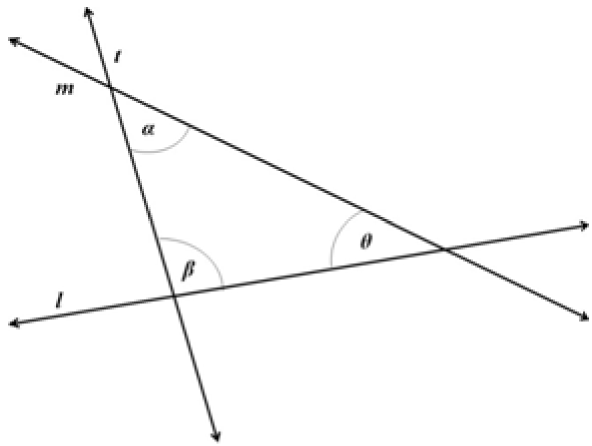

Figure 4.Proving parallel postulate

<u>Case I:</u> When α and β are both acute.

Let α = 90˚ - *x* and β = 90˚ - *y* [for some positive constants *x* and *y*]

$\therefore \theta = 180^\circ - \{(90^\circ - x) + (90^\circ - y)\} = 180^\circ - \{180^\circ - x - y\} = x + y \in \mathbf{Z}^+$

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Hence, the lines would intersect in this case.

Case II: When one is acute and another a right angle.

Let $\alpha = 90^\circ - x$ and $\beta = 90^\circ$ [for some positive constant x]

$\therefore \theta = 180^\circ - \{(90^\circ - x) + 90^\circ\} = 180^\circ - \{180^\circ - x\} = x \in \mathbf{Z}^+$

Hence, the lines would intersect in this case.

Case III: When one is acute and another obtuse.

Let $\alpha = 90^\circ - x$ and $\beta = 90^\circ + y$ [for some positive constants x and y]

$\therefore \theta = 180^\circ - \{(90^\circ - x) + (90^\circ + y)\} = 180^\circ - \{180^\circ - x + y\} = x - y \in \mathbf{Z}^+$ iff $x > y$

Hence, the lines would intersect in this case iff the difference between α and a right angle is less than that between β and a right angle.

Case IV: When both are right angles i.e. $\alpha = \beta = 90^\circ$.

$\therefore \theta = 180^\circ - (90^\circ + 90^\circ) = 180^\circ - 180^\circ = 0^\circ$ [doesn't belong to the set of positive integers]

Hence, the lines won't intersect in this case.

Case V: When one is obtuse and another a right angle.

Let $\alpha = 90^\circ + x$ and $\beta = 90^\circ$ [for some positive constant x]

$\therefore \theta = 180^\circ - \{(90^\circ + x) + 90^\circ\} = 180^\circ - \{180^\circ + x\} = -x$ [doesn't belong to the set of positive integers]

Hence, the lines won't intersect in this case.

Case VI: When both are obtuse.

Let $\alpha = 90^\circ + x$ and $\beta = 90^\circ + y$ [for some positive constants x and y]

$\therefore \theta = 180^\circ - \{(90^\circ + x) + (90^\circ + y)\} = 180^\circ - \{180^\circ + x + y\} = -(x + y)$ [doesn't belong to the set of positive integers]

Hence, the lines won't meet in this case.

Thus, we see that, the lines would meet if and only if $\alpha + \beta < 180^\circ$.

We may conclude the same in an alternate way as below:

Proof. Let us assume that we have four points A, A′, B and B′ on a plane. By Euclid's first postulate, we may draw two straight line segments AA′ and BB′. Treating the line segments

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as straight lines, according to the second postulate, we may extend the straight lines infinitely in both directions. Suppose l be a line intersecting both the lines. If the lines would meet, there will be a value of θ.

$\therefore \alpha + \beta + \theta = 180^\circ \Rightarrow \alpha + \beta = 180^\circ - \theta < 180^\circ \Rightarrow \alpha + \beta < 180^\circ$

Hence, we see that whenever the lines intersect, $\alpha + \beta < 180^\circ$.

Remark. If $\alpha + \beta >$ two right angles, then the lines would intersect in the opposite side of the interior angles taken in consideration. If $\alpha + \beta$ equals two right angles, then the lines won't intersect in either side.

Proof.

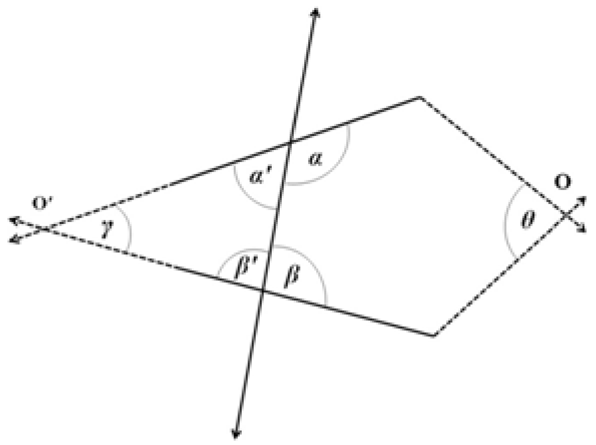

Figure 5.Proving that if the lines move away from each other in one side, they intersect in the other

Let $\alpha + \beta = 180^\circ + x > 180^\circ$ [for some positive constant x]. Clearly θ in this case is $-x$. We name the supplement of α and β as α′ and β′ respectively (as in Figure 5).

Now, $\alpha' = 180^\circ - \alpha$ and $\beta' = 180^\circ - \beta$; then,

γ (as in Figure 5) $= 180^\circ - \{(180^\circ - \alpha) + (180^\circ - \beta)\} \Rightarrow 180^\circ - \{360^\circ - (\alpha + \beta)\} \in \mathbf{Z}^+$ [$\because \alpha + \beta > 180^\circ$]

Hence, there is a value of γ which implies that the lines would meet at O′.

Let $\alpha + \beta = 180^\circ$. Clearly θ in this case equals 0. Then, $\gamma = 180^\circ - \{(180^\circ - \alpha) + (180^\circ - \beta)\} \Rightarrow 180^\circ - \{360^\circ - (\alpha + \beta)\} = 0$ doesn't belong to $\mathbf{Z}^+$ [$\because \alpha + \beta = 180^\circ$].

Hence, the lines won't meet on either side.

Now, we move on to prove the propositions of parallel postulate with the postulate of collinearity and parallelism.

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Proving the propositions of parallel postulate by postulate of collinearity and parallelism

As we proved that the sum of all interior angles of any triangle equals 180˚ with the postulate of collinearity and parallelism, we ignore the theorems that are provable by the triangle postulate in this section.

Proposition. *A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.*

Suppose AB||CD and EF be the transversal line intersecting AB and CD at X and Y respectively. We would now prove each result in the proposition separately.

To prove. ∠AXF = ∠DYE and ∠FXB = ∠EYC

Proof. Suppose $XO_1 \perp CD$ and $YO_2 \perp AB$. By postulate of collinearity and parallelism, $XO_1 = YO_2$. Clearly, $\Delta XO_2Y \cong \Delta YO_1X$ [By RHS criterion].

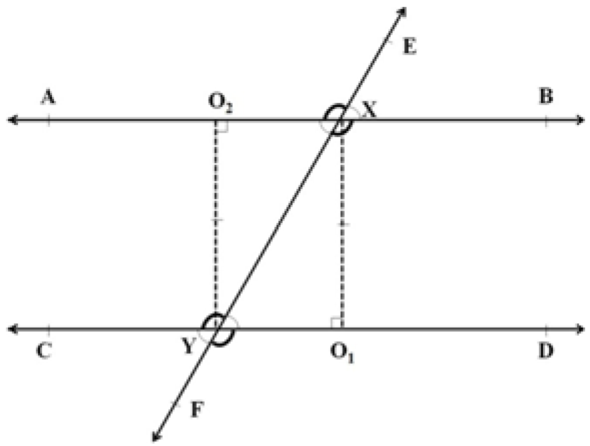

Figure 6.Here, AB||CD and EF is the transversal

∴ ∠AXF = ∠DYE

Similarly, ∠FXB = 180˚ - ∠AXF and ∠EYC = 180˚ - ∠DYE

∴ ∠ FXB = ∠EYC [∵ ∠AXF = ∠DYE]

To prove. ∠EXB = ∠AXF and ∠FXB = ∠EXA; ∠EYD = ∠FYC and ∠CYE

Proof. From Figure 6, we get;

∠EXB + ∠FXB = 180˚ = ∠EXB + ∠EXA

⇒ ∠EXB + ∠FXB = ∠FXB + ∠EXA

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⇒ ∠FXB = ∠EXA

Also, ∠EXA + ∠AXF = 180˚ = ∠EXA + ∠EXB

⇒ ∠EXA = ∠AXF = ∠EXA + ∠EXB

⇒ ∠AXF = ∠EXB

In similar way, we can prove, ∠EYD = ∠FYC and ∠CYE = ∠FYD.

To prove. ∠AXF + ∠EYC = 180◦ and ∠BXF + ∠DYE = 180˚

Proof. As we proved earlier, the considered alternate angles are equal to one another. This means, ∠EXC = ∠BXF.

∴ ∠AXF + ∠EYC = ∠AXF + ∠BXF = 180˚ [linear pair]

Similarly, ∠BXF + ∠DYE = ∠EYC + ∠DYE = 180˚ [linear pair]

Hence, we could prove the proposition with postulate of collinearity and parallelism.

Proposition. *Straight lines parallel to the same straight line are also parallel to one another.*

Proof. Let us assume a system of three lines AB, CD and EF. Notably AB||CD and CD||EF. We have to prove AB||EF.

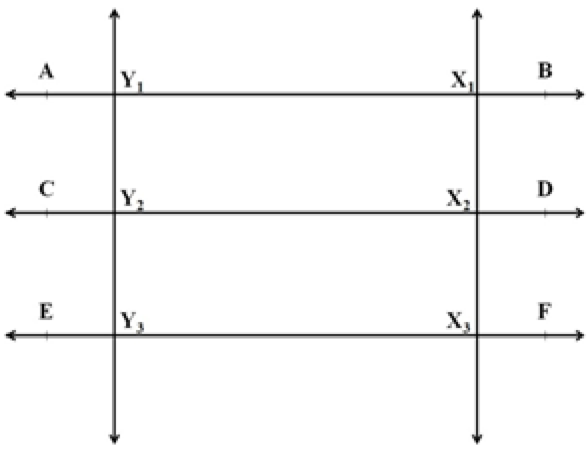

Figure 7.Here, AB||CD and CD||EF

We mark points X_1, X_2, X_3, Y_1, Y_2, Y_3 accordingly as in Figure 7 such that $X_1X_2 \perp AB$, $Y_1Y_2 \perp AB$, $X_2X_3 \perp EF$ and $Y_2Y_3 \perp EF$. By postulate of collinearity and parallelism, $X_1X_2 = Y_1Y_2$ and $X_2X_3 = Y_2Y_3$.

Clearly, $\angle Y_1Y_2X_2 = \angle Y_3Y_2X_2 = \angle X_1X_2Y_2 = \angle X_3X_2Y_2 = 90^\circ$ [∵ by the postulate by Euclid, all right angles are congruent]

This means Y_1Y_2 and X_1X_3 are straight lines. [∵ $\angle Y_1Y_2Y_3 = \angle X_1X_2X_3 = 180^\circ$]

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Let, $X_1X_2 = Y_1Y_2 = x$ and $X_2X_3 = Y_2Y_3 = y$.

Now, $X_1X_3 = x + y = Y_1Y_3$.

∴ By postulate of collinearity and parallelism, AB||EF.

Proposition. *In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.*

Proof. We have already proved the triangle sum property with the postulate of collinearity and parallelism. We would use this result to prove the above proposition.

Let α, β and γ be the interior angles formed at vertices A, B and C respectively and θ be the exterior angle at vertex C of ΔABC. We know, $\alpha + \beta + \gamma = 180^\circ$ [as we proved earlier]. Now,

$\alpha + \beta + \gamma = 180^\circ \Rightarrow \alpha + \beta = 180^\circ - \gamma = \theta$ [$\because \theta + \gamma = 180^\circ$] $\Rightarrow \alpha + \beta = \theta$

So, we proved that the exterior angle equals the sum of opposite interior angles of a triangle.

Similarly, we can prove all the propositions and theorems derived from parallel postulate with other postulates of Euclid and the postulate of collinearity and parallelism. This shows the postulate of collinearity and parallelism worthy to be a substitute of the parallel postulate.

Now, we prove the Playfair's axiom which is equivalent to the parallel postulate as a theorem with the postulate of collinearity and parallelism.

Proving Playfair's axiom

The Playfair's axiom states that *in a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point*.

Proof. Let us assume a straight line through points A_0 and B_0. Assume a point O not on the line. Except the line through O which is perpendicular to A_0B_0, we consider all other possible lines through the same. Let the lines be denoted by points A_1, B_1, A_2, B_2, ... , A_n, B_n such that all the considered points are collinear (for our convenience to prove the statement). We assume all the lines i.e. A_1B_1, A_2B_2, A_3B_3, ... , A_nB_n be parallel to A_0B_0. Also, consider a point X such that $OX \perp A_0B_0$.

By postulate of collinearity and parallelism,

$OX = A_0A_1 = A_0A_2 = A_0A_3 = \ldots = A_0A_n = B_0B_1 = B_0B_2 = \ldots = B_0B_n$

$\Rightarrow A_0A_1 = A_0A_2 = A_0A_3 = \ldots = A_0A_n = B_0B_1 = B_0B_2 = \ldots = B_0B_n$

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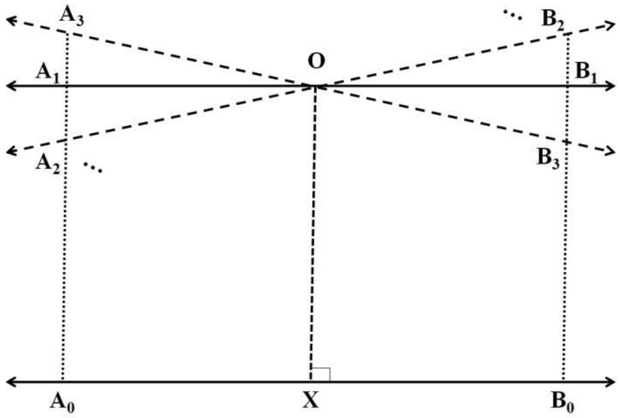

Figure 8.Proving Playfair's axiom

∵ $A_1, A_2, A_3, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are on two distinct lines,

∴ The above relation implies that $A_1, A_2, A_3, \ldots, A_n$ and $B_1, B_2, B_3, \ldots, B_n$ coincide at two respective points. [We assume the line through $A_1, A_2, A_3, \ldots, A_n$ to A_0 and $B_1, B_2, B_3, \ldots, B_n$ to B_0 to be perpendicular to A_0B_0.]

Hence, we proved that there is exactly one line through O which is parallel to A_0B_0.

Conclusion

So, we proved that the postulate of collinearity and parallelism is worthy to substitute the then stated parallel postulate by proving the theorems and propositions derived from the same. We may conclude the parallel postulate and the Playfair's axiom to be theorems provable by the postulate of collinearity and parallelism.

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