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FIXED POINT THEOREMS TO SYSTEMS OF LINEAR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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ABSTRACT

In this paper, the systems of linear Volterra integral equation of second order were investigated using fixed point theorems. It was found out that there is an existence and a unique common solution of two linear systems of Volterra equations of second kind:

$$y(x) = f_1(x) + \lambda \int_a^x K_1(x,t) F_1(y(t))dt$$
$$y(x) = f_2(x) + \lambda \int_a^x K_2(x,t) F_2(y(t))dt, \quad \text{where} \quad x \in [0,T],$$

$\lambda \in IR$, f_1, f_2, K_1, K_2, F_1 *and* F_2 *are given continuous functions, while* y *is unknown function to be determined.*

1. INTRODUCTION

In Mathematics, the Volterra integral equations are special type of integral equations while integral equations are equations in which an unknown function appears under an integral sign. An Italian mathematician and physicist Professor Vito Volterra (1883) developed a theory of functional which led to integral and integro-differential equations. His papers on integral equations which is now called Volterra integral equations appeared in 1896 which later studied by Traian Lalescu (1908) during his thesis-'Sur les equations de Volterra' written under the direction of Emile Picard. In 1911, Lalescu wrote his first book ever on integral equations. Volterra's contributions were relevant to mathematical biology and integral equations (1930) being one of the founders of functional analysis and principally reiterating and developed the work of Pierre François Verhulst (1804) which later led to Lotka–Volterra equations (1920).

Volterra integral equations find application in demography, the study of viscoelastic materials and in actuarial science through the renewal equation. The Renewal Volterra Integral Equation is given as

$$y(x) = f(x) + \lambda \int_0^x K(x-t) F(y(t))dt, t \geq 0$$

where ($K(x,t) = K(x-t)$ called a Convolution kernel: it depends only on the difference $x-t$ of the variables x and t) arises in the mathematical modeling of Renewal processes.

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Examples include: model with single commodity (a single investment policy that is continually renewed and a single depreciation policy) and model of age-structured population in which individuals die and new individuals are added (born). Also, for a population growth model, $F(y(t))$ is the number of members added to the population (in unit time) when the population size is $y(t)$, $K(x)$ represents the probability that a member of the population survives at age x and the function $f(x)$ represents the number of members who are already present at $x=0$ and who are still alive at time $x>0.$ A stability study criterion for Volterra equation can be based on the contraction mapping principle- Burton (2005). The criterion has been significantly used. In 1823 Abel suggested a generalization of the tautochrone problem, the solution of which included the solution of an integral equation that was later known as an integral first form equation and in 1837 Liouville demonstrated that it would be possible to determine a particular solution of a linear differential equation of the second order by solving an integral equation of another kind, called the integral equation of the second type. At first, but gradually, the ripple of mathematical interest in these investigations increased.

Recently, however, inspired by Volterra, Fredholm, and Hilbert's work in the time between 1896 and the present, what seemed at first only a ripple has developed into a formidable wave that bids fair to take the integral equation theory to a position next to the most important mathematical disciplines.

2. RELATED LITERATURE REVIEWS

Some scientists have looked at Volterra equations in some ways including analytical methods. The following are existing literatures.

Banas and Rzepka (2003) focused on the implementation of a measure of non-compactness in the analysis of asymptotic stability. They used the Darbo method fixed point theorem technique which is associated with noncompactness dimensions to develop a claim of existence for some functional integral equations.

Burton (2003) studied an integral equation's stability by comparing two fixed point theory approches and the theory of Liapunov. The result showed that the direct approach of Liapunov is ineffective compared to the fixed point theorem in maintaining stability.

Burton (2004) studied the nature of an integral equation of Banas and Rzepka and its asymptotic stability. He used the fixed-point principle approach and noncompactness measurements. His primary finding was that the calculations of noncompactness could be prevented and the presence and stability under weaker conditions was confirmed.

Burton (2005) used fixed points to test the consistency of the Volterra equation. He established the existence of the linear problem's periodic solution.

Chen *et. al* (2013) used fixed-point methods to analyze stability effects for nonlinear functional differential equations. In a coherent context, their work expanded and improved a number of recent findings of stability for nonlinear functional differential equations.

Jin and Lee (2014) used a fixed-point approach to determine the consistency of the functional equation to test a mixed form additive and quadratic functional equation. The key finding was that the mixed-type additive and quadratic functional equation can demonstrate the generalized stability of the Hyers-Ulam.

3. MODEL FORMULATION

Following Rahman (2007), a typical form of Volterra integral equation in $y(x)$ is of the form

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$$y(x) = f(x) + \lambda \int_{\beta(x)}^{\alpha(x)} K(x,t)y(t)dt \tag{3.1}$$

where $K(x,t)$ is known as the kernel of the integral equation and $\alpha(x)$ and $\beta(x)$ are also called the limit of integration, while λ is called a constant parameter.

Mostly, Volterra integral equation can be given in a more standard form as

$$y(x)g(x) = f(x) + \lambda \int_{a}^{x} K(x,t)y(t)dt \tag{3.2}$$

The unknown function $y(x)$ appears linearly under the integral sign. (3.2) is called Volterra integral equation of the second kind if the function $g(x) = 1$, thus:

$$y(x) = f(x) + \lambda \int_{a}^{x} K(x,t)y(t)dt \tag{3.3}$$

4. METHODS

Lemma 4.1 (Waleed *et. al* 2018) Given that (X,d) is a complex metric space and $\{u_n\}_{n=1}^{\infty}$ a sequence in X. If $\{u_n\}_{n=1}^{\infty}$ is not Cauchy in X, then $\exists \varepsilon_0 > 0$ and two subsequences $\{u_{n(k)}\}_{n=1}^{\infty}$ and $\{u_{m(k)}\}_{m=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$: $k < m(k) < n(k) < m(k+1), |d(u_n,u_m)| \geq \varepsilon_0, |d(u_{m(k)},u_{n(k)-1})| < \varepsilon_0$.

Proof for Lemma 4.1 Since that $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence *iff* $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathrm{N} :| d(u_n,u_m)| \leq \varepsilon, \forall n,m \geq n_\varepsilon$.

Suppose for contradiction, then $\exists \varepsilon_0 > 0, \forall N \in \mathrm{N}, \exists n,m \geq N, | d(u_n,u_m)| \geq \varepsilon_0$.

Let $N = 2.$ Then $\exists n_1, m_1 \geq 2 :| d(u_{n_1},u_{m_1})| \geq \varepsilon_0$. Let $\min\{n_1,m_1\} := m(1)$ and consider $| d(u_{m(1)},u_{m(1)+1})|, | d(u_{m(1)},u_{m(1)+2})|, \ldots, | d(u_{m(1)},u_{\max}\{n_1,m_1\})|$.

Since $| d(u_{m(1)},u_{\max}\{n_1,m_1\})| = | d(u_{n_1},u_{m_1})| \geq 0,$

thus, $n(1) \in \{m(1)+1, m(1)+2, \ldots, \max\{n_1,m_1\}\} :| d(u_{m(1)},u_{n(1)})| \geq \varepsilon_0, | d(u_{m(1)},u_i)| < \varepsilon_0$, $\forall i \in \{m(1)+1, m(1)+2, \ldots, n(1)-1\}$.

In particular, $| d(u_{m(2)},u_{n(2)-1})| < \varepsilon_0$. Recursively, there are two subsequences $\{u_{n(k)}\}_{n=1}^{\infty}$ and $\{u_{m(k)}\}_{m=1}^{\infty}$ also such that

$$k < m(k) < n(k) < m(k+1), | d(u_{n(k)},u_{m(k)})| \geq \varepsilon_0, | d(u_{m(k)},u_{n(k)-1})| < \varepsilon_0, \forall k \in \mathrm{N}.$$

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Theorem 4.2.1 (Waleed *et. al* 2018) Let (T,Q) be a pair of self-mappings defined on complex metric space (X,d) and giving the mappings $\mu_1,\mu_2 : C_+ \to [0,1)$ such that $\mu_1+\mu_2 \in \Gamma$. Assume that $TX \cup QX$ is complete subspace of X and for all $u,v \in X$,

$$d(Tu,Qv) \precsim \mu_1(d(u,v))d(u,v) + \mu_2(d(u,v))\frac{d(Tu,u)d(Qv,v)}{1+d(Tu,Qv)}. \tag{3.4}$$

Then, the pair (T,Q) has a unique fixed point.

Proof 4.2.1 Let u_0 be an arbitrary element of X. Construct a sequence $\{u_n\}_{n=0}^{\infty}$ in $TX \cup QX$ as follows:

$$u_{2n+1} = Tu_{2n},\ u_{2n+2} = Qu_{2n+1}, n = 0,1,2,... \tag{3.5}$$

Now, suppose $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence. For all $n \geq 0$, using (3.4) and (3.5) to get:

$$\begin{aligned}
&d(u_{2n+1},u_{2n+2}) = d(Tu_{2n},Qu_{2n+1})\\
&\precsim \mu_1(d(u_{2n},u_{2n+1}))d(u_{2n},u_{2n+1}) + \mu_2(d(u_{2n},u_{2n+1}))\frac{d(Tu_{2n},u_{2n})d(Qu_{2n+1},u_{2n+1})}{1+d(Tu_{2n},Qu_{2n+1})}\\
&= \mu_1(d(u_{2n},u_{2n+1}))d(u_{2n},u_{2n+1}) + \mu_2(d(u_{2n},u_{2n+1}))\frac{d(u_{2n+1},u_{2n})d(u_{2n+2},u_{2n+1})}{1+d(u_{2n+1},u_{2n+2})}\\
&\Rightarrow |d(u_{2n+1},u_{2n+2})| \precsim \mu_1(d(u_{2n},u_{2n+1}))|d(u_{2n},u_{2n+1})| + \mu_2(d(u_{2n},u_{2n+1}))\left|\frac{d(u_{2n+1},u_{2n})d(u_{2n+2},u_{2n+1})}{1+d(u_{2n+1},u_{2n+2})}\right|\\
&= \mu_1(d(u_{2n},u_{2n+1}))|d(u_{2n},u_{2n+1})| + \mu_2(d(u_{2n},u_{2n+1}))|d(u_{2n+1},u_{2n})|.\left|\frac{d(u_{2n+2},u_{2n+1})}{1+d(u_{2n+1},u_{2n+2})}\right|\\
&\precsim \mu_1(d(u_{2n},u_{2n+1}))|d(u_{2n},u_{2n+1})| + \mu_2(d(u_{2n},u_{2n+1}))|d(u_{2n+1},u_{2n})|.1\\
&= \mu_1(d(u_{2n},u_{2n+1}))|d(u_{2n},u_{2n+1})| + \mu_2(d(u_{2n},u_{2n+1}))|d(u_{2n+1},u_{2n})|\\
&= |d(u_{2n},u_{2n+1})|\left[\mu_1(d(u_{2n},u_{2n+1})) + \mu_2(d(u_{2n},u_{2n+1}))\right]\\
&= |d(u_{2n},u_{2n+1})|\left[d(u_{2n+1},u_{2n})(\mu_1+\mu_2)\right]\\
&= (\mu_1+\mu_2)(d(u_{2n+1},u_{2n}))|d(u_{2n+1},u_{2n})|\\
&\therefore d(u_{2n+1},u_{2n+2}) \precsim (\mu_1+\mu_2)(d(u_{2n+1},u_{2n}))|d(u_{2n+1},u_{2n})|
\end{aligned} \tag{3.6}$$

Similarly, following Sintunavarant *et al.* (2013): $\forall n \geq 0,$ we get:

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$$
\begin{aligned}
&d(u_{2n+2},u_{2n+3})=d(u_{2n+3},u_{2n+2})\\
&=d(Tu_{2n+2},Qu_{2n+1})\\
&\precsim \mu_1(d(u_{2n+2},u_{2n+1}))d(u_{2n+2},u_{2n+1})+\frac{\mu_2(d(u_{2n+2},u_{2n+1}))d(u_{2n+2},Qu_{2n+2})d(u_{2n+1},Qu_{2n+1})}{1+d(u_{2n+2},u_{2n+1})}\\
&=\mu_1(d(u_{2n+2},u_{2n+1}))d(u_{2n+2},u_{2n+1})+\mu_2(d(u_{2n+2},u_{2n+1}))d(u_{2n+2},Qu_{2n+2}).\frac{d(u_{2n+1},Qu_{2n+1})}{1+d(u_{2n+2},u_{2n+1})}\\
&\Rightarrow |d(u_{2n+2},u_{2n+3})| \precsim \mu_1(d(u_{2n+2},u_{2n+1}))\,|d(u_{2n+2},u_{2n+1})|\\
&+\mu_2(d(u_{2n+2},u_{2n+1}))\,|d(u_{2n+2},Qu_{2n+2})|.\left|\frac{d(u_{2n+1},Qu_{2n+1})}{1+d(u_{2n+2},u_{2n+1})}\right|\\
&\precsim \mu_1(d(u_{2n+2},u_{2n+1}))\,|d(u_{2n+2},u_{2n+1})|+\mu_2(d(u_{2n+2},u_{2n+1}))\,|d(u_{2n+2},Qu_{2n+2})|.1\\
&=|d(u_{2n+2},u_{2n+1})|\left[\mu_1(d(u_{2n+2},u_{2n+1}))+\mu_2(d(u_{2n+2},u_{2n+1}))\right]\\
&=(\mu_1+\mu_2)(d(u_{2n+2},u_{2n+1}))\,|d(u_{2n+2},u_{2n+1})|
\end{aligned}
$$

$$\therefore d(u_{2n+2},u_{2n+3}) \precsim (\mu_1+\mu_2)(d(u_{2n+2},u_{2n+1}))\,|d(u_{2n+2},u_{2n+1})| \tag{3.7}$$

Following (3.6) and (3.7) to get:

$$d(u_n,u_{n+1}) \precsim (\mu_1+\mu_2)(d(u_{n-1},u_n))\,|d(u_{n-1},u_n)| \precsim |d(u_{n-1},u_n)|\ \forall n\in N. \tag{3.8}$$

$\Rightarrow \{|d(u_n,u_{n+1})|\}_{n=0}^{\infty}$ is monotone non-decreasing and bounded above.

$\therefore \{|d(u_n,u_{n+1})|\}_{n=0}^{\infty} \to r, r\geq 0.$

Now, claiming that $r=0$, otherwise, suppose for contradiction that $r>0$. In (3.8), taking $n\to\infty$, to have $|(\mu_1+\mu_2)(d(u_{n-1},u_n))| \to 1$

Since $(\mu_1+\mu_2)\in\Gamma$, $\Rightarrow \{|(d(u_{n-1},u_n))\}\to 0$, which is a contradiction.

$\therefore |d(u_n,u_{n+1})| \to 0$ as $n\to\infty$.

Equivalently,

$$\lim_{n\to\infty.} d(u_n,u_{n+1})=0. \tag{3.9}$$

Next, from (3.9), it is enough to show that $\{u_{2n(k)}\}_{n=1}^{\infty}$ is Cauchy sequence. Thus, suppose for contradiction that $\{u_{2n(k)}\}_{n=1}^{\infty}$ is not Cauchy sequence. Hence, from Lemma (3.1), $\varepsilon_0>0$ and given $\{u_{2n(k)}\}_{n=1}^{\infty}$ and $\{u_{2m(k)}\}_{m=1}^{\infty}$ be the two subsequences such that

$$k<m(k)<n(k)<m(k+1), |d(u_{2n(k)},u_{2m(k)})| \geq \varepsilon_0, |d(u_{2n(k)},u_{2m(k)-1})|<\varepsilon_0, \forall k\in\mathrm{N}. \tag{3.10}$$

But from (3.10):

$$\begin{aligned}
&\varepsilon_0 \leq | d(u_{2n(k)}, u_{2m(k)}) | \\
&\leq | d(u_{2n(k)}, u_{2m(k)-2}) | + | d(u_{2m(k)-2}, u_{2m(k)-1}) | + | d(u_{2m(k)-1}, u_{2m(k)}) | \\
&< \varepsilon_0 + | d(u_{2m(k)-2}, u_{2m(k)-1}) | + | d(u_{2m(k)-1}, u_{2m(k)}) | .
\end{aligned} \tag{3.11}$$

Setting $k \to \infty$, thus (3.11) becomes :

$$\begin{aligned}
&\varepsilon_0 \leq | d(u_{2n(k)}, u_{2m(k)}) | \leq \varepsilon_0 \\
&\Leftrightarrow | d(u_{2n(k)}, u_{2m(k)}) | = \varepsilon_0 .
\end{aligned} \tag{3.12}$$

Moreover, $$\begin{aligned}
&| d(u_{2n(k)}, u_{2m(k)}) | \leq | d(u_{2n(k)}, u_{2m(k)+1}) | + | d(u_{2m(k)+1}, u_{2m(k)} | \\
&\leq | d(u_{2n(k)}, u_{2m(k)}) | + | d(u_{2m(k)}, u_{2m(k)+1}) | + | d(u_{2m(k)+1}, u_{2m(k)}) |
\end{aligned}$$

Setting $k \to \infty$, thus (3.12) becomes :

$$\begin{aligned}
&\varepsilon_0 \leq | d(u_{2n(k)}, u_{2m(k)+1}) | \leq \varepsilon_0 \\
&\Leftrightarrow | d(u_{2n(k)}, u_{2m(k)+1}) | = \varepsilon_0 .
\end{aligned} \tag{3.13}$$

Next, considering

$$\begin{aligned}
&d(u_{2n(k)}, u_{2m(k)+1}) \precsim d(u_{2n(k)}, u_{2n(k)+1}) + d(u_{2n(k)+1}, u_{2m(k)+2}) + d(u_{2n(k)+2}, u_{2m(k)+1}) \\
&= d(u_{2n(k)}, u_{2n(k)+1}) + d(Tu_{2n(k)}, Qu_{2m(k)+1}) + d(u_{2m(k)+2}, u_{2m(k)+1}) \\
&\precsim d(u_{2n(k)}, u_{2n(k)+1}) + d(u_{2m(k)+2}, u_{2m(k)+1}) + \mu_1(d(u_{2n(k)}, u_{2m(k)+1})).d(u_{2n(k)}, u_{2m(k)+1}) \\
&+ \mu_2(d(u_{2n(k)}, u_{2m(k)+1})) \frac{d(Tu_{2n(k)}, u_{2n(k)}) d(Qu_{2m(k)+1}, u_{2m(k)+1})}{1 + d(Tu_{2n(k)}, Qu_{2m(k)+1})} \\
&\precsim d(u_{2n(k)}, u_{2n(k)+1}) + d(u_{2m(k)+2}, u_{2m(k)+1}) + \mu_1(d(u_{2n(k)}, u_{2m(k)+1})).d(u_{2n(k)}, u_{2m(k)+1}) \\
&+ \mu_2(d(u_{2n(k)}, u_{2m(k)+1})) \frac{d(u_{2n(k)+1}, u_{2n(k)}) d(u_{2m(k)+2}, u_{2m(k)+1})}{1 + d(u_{2n(k)+1}, u_{2m(k)+2})} \\
&\Rightarrow d(u_{2n(k)}, u_{2m(k)+1}) \leq | d(u_{2n(k)}, u_{2n(k)+1}) | + | d(u_{2m(k)+2}, u_{2m(k)+1}) | \\
&+ \mu_1(d(u_{2n(k)}, u_{2m(k)+1})). | d(u_{2n(k)}, u_{2m(k)+1}) | \\
&+ \mu_2(d(u_{2n(k)}, u_{2m(k)+1})) \left| \frac{d(u_{2n(k)+1}, u_{2n(k)}) d(u_{2m(k)+2}, u_{2m(k)+1})}{1 + d(u_{2n(k)+1}, u_{2m(k)+2})} \right|
\end{aligned}$$

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$$\begin{aligned}
&\leq |d(u_{2n(k)},u_{2n(k)+1})|+|d(u_{2m(k)+2},u_{2m(k)+1})| \\
&+(\mu_1+\mu_2)(d(u_{2n(k)},u_{2m(k)+1}))|d(u_{2n(k)},u_{2m(k)+1)}| \\
&+\mu_2(d(u_{2n(k)},u_{2m(k)+1}))|\frac{d(u_{2n(k)+1},u_{2n(k)})d(u_{2m(k)+2},u_{2m(k)+1})}{1+d(u_{2n(k)+1},u_{2m(k)+2})}| \\
&\leq |d(u_{2n(k)},u_{2n(k)+1})|+|d(u_{2m(k)+2},u_{2m(k)+1})| \\
&+|d(u_{2m(k)+2},u_{2m(k)+1})|+|\frac{d(u_{2n(k)+1},u_{2n(k)})d(u_{2m(k)+2},u_{2m(k)+1})}{1+d(u_{2n(k)+1},u_{2m(k)+2})}|
\end{aligned} \tag{3.14}$$

Setting $k \to \infty$, thus (3.14) becomes :

$$\begin{aligned}
&\varepsilon_0 \leq (\lim_{k\to\infty}(\mu_1+\mu_2)(d(u_{2n(k)},u_{2m(k)+1})))\varepsilon_0 \leq \varepsilon_0 \\
&\Leftrightarrow 1 \leq (\lim_{k\to\infty}(\mu_1+\mu_2)(d(u_{2n(k)},u_{2m(k)+1}))) \leq 1 \\
&\Leftrightarrow \lim_{k\to\infty}(\mu_1+\mu_2)(d(u_{2n(k)},u_{2m(k)+1})=1 \\
&\therefore \lim_{k\to\infty}|d(u_{2n(k)},u_{2m(k)+1}|=0,(\mu_1+\mu_2)\in\Gamma
\end{aligned} \tag{3.15}$$

(3.15) contradicts (3.13) as $\varepsilon_0 > 0$. Hence, $\{u_n\}_{n=1}^{\infty}$ is Cauchy sequence. Finally, by the completeness of $TX \cup QX, \exists u \in TX \cup QX : \{u_n\}_{n=1}^{\infty} \to u.$

Next, to prove for uniqueness of the common fixed point of (T, Q).

Suppose that $Tu = u.$

By contradiction, suppose that $Tu \neq u$, then $|d(u,Tu)|> 0$.

But,

$$\begin{aligned}
&d(u,Tu) \precsim d(u,u_{2n+2})+d(u_{2n+2},Tu) \\
&= d(u,u_{2n+2})+d(Tu,Qu_{2n+1}) \\
&\precsim d(u,u_{2n+2})+\mu_1(d(u,u_{2n+1}))d(u,u_{2n+1}) \\
&+\mu_2(d(u,u_{2n+1}))\frac{d(Tu,u)d(Qu_{2n+1},u_{2n+1})}{1+d(Tu,Qu_{2n+1})}
\end{aligned}$$

$$\begin{aligned}
&= d(u,u_{2n+2})+\mu_1(d(u,u_{2n+1}))d(u,u_{2n+1}) \\
&+\mu_2(d(u,u_{2n+1}))\frac{d(Tu,u)d(u_{2n+2},u_{2n+1})}{1+d(Tu,u_{2n+2})}
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\therefore |d(u,Tu)| \leq |d(u,u_{2n+2})|+|\mu_1(d(u,u_{2n+1}))||d(u,u_{2n+1})| \\
&+|\mu_2(d(u,u_{2n+1}))||\frac{d(Tu,u)d(u_{2n+2},u_{2n+1})}{1+d(Tu,u_{2n+2})}| \\
&\leq |d(u,u_{2n+2})|+|d(u,u_{2n+1})|+|\frac{d(Tu,u)d(u_{2n+2},u_{2n+1})}{1+d(Tu,u_{2n+2})}|
\end{aligned}$$

As $n \to \infty, |d(u,Tu)| \leq 0,$ this gives a contradiction.

$\Rightarrow Tu = u.$ In the same way, suppose that $Qu = u.$ Hence, $Tu = Qu = u \Leftrightarrow u$ is a common fixed point of the pair (T,Q).

Next, to show that u is unique. Suppose for contradiction that u' is also a common fixed point of the pair (T,Q).

Following (3.16):

$$d(u,u') = d(Tu,Qu') \preceq \mu_1(d(u,u')).d(u,u')$$
$$+ \mu_2(d(u,u'))\frac{d(Tu,u)d(Qu',u')}{1+d(Tu,Qu')}$$
$$= \mu_1(d(u,u'))d(u,u').$$
$$\Rightarrow |d(u,u')| \leq \mu_1(d(u,u'))\,|d(u,u')| < |d(u,u')|$$

$\Leftrightarrow |d(u,u')| < |d(u,u')|$. This leads to a contradiction.

Hence, u is a unique common fixed point of the pair (T,Q).

5. RESULTS

In this part, theorem (4.2.1) is used to prove the existence and uniqueness of a common solution of the following system of linear Volterra Integral Equations of the second kind:

$$\begin{aligned} y(x) &= f_1(x) + \lambda\int_a^x K_1(x,t)F_1(y(t))dt \\ y(x) &= f_2(x) + \lambda\int_a^x K_2(x,t)F_2(y(t))dt \end{aligned} \tag{5.1}$$

where $x \in [0,T], \lambda \in IR, f_1, f_2, K_1, K_2$ and F_1 and F_2 are given continuous functions and y is an unknown function to be determined.

The following symbols are used for simplicity:

$$G_i(y(x)) = \int_a^x K_i(x,t)F_i(y(t))dt,$$
$$\Gamma_{uv}(x) = \| u(x) - v(x) \| e^i,$$
$$\Lambda_{uv}(x) = \| f_1(x) + G_1(u(x)) - u(x) \| e^i,$$
$$\Upsilon_{uv}(x) = \| f_2(x) + G_2(v(x)) - v(x) \| e^i,$$
$$\Omega_{uv}(x) = \| f_1(x) + G_1(u(x)) - f_2(x) - G_2(v(x)) \| e^i$$

with $X = C([a,x], R)$ is the space of all real valued continuous functions defined on $[a,x]$.

Suppose two mappings on X are defined as follows:

$$Ty(x) = f_1(x) + G_1(y(x)) = f_1(x) + \int_a^x K_1(x,t)F_1(y(t))dt \tag{5.2}$$

$$Qy(x) = f_2(x) + G_2(y(x)) = f_2(x) + \int_a^x K_1(x,t)F_1(y(t))dt \tag{5.3}$$

The system in (5.1) has a unique common solution if and only if, the pair (T,Q) in (5.2) and (5.3) has a unique fixed point.

6. DISCUSSION OF RESULTS

Theorem 6.1 The system (5.1) of linear Volterra Integral Equations of the second order has a unique common fixed point if

i. $\exists \mu_1, \mu_2 : C_+ \to [0,1) : \mu_1 + \mu_2 \in \Gamma.$

ii. $\forall u, v \in X, x \in [0,T]$

then, $\Omega_{uv}(x) \precsim \mu_1(\max_{x\in[0,T]} \Gamma_{uv}(x))\Gamma_{uv}(x) + \mu_2(\max_{x\in[0,T]} \Gamma_{uv}(x)) \dfrac{\Lambda_{uv}(x)\Upsilon_{uv}(x)}{1+\max_{x\in[0,T]} \Omega_{uv}(x)} \quad x \in [0,T]$

Proof 6.1 Defining a map $d : X \times X \to C_+$ as follows: $d(u,v) = \max_{x\in[0,T]} \| u(t) - v(t) \| e^i$. Then (X,d) is a complete complex valued metric space.

Next, from (ii): $\forall u, v \in X, x \in [0,T]$,

$$\Omega_{uv}(x) \precsim \mu_1(\max_{x\in[0,T]} \Gamma_{uv}(x))\Gamma_{uv}(x) + \mu_2(\max_{x\in[0,T]} \Gamma_{uv}(x)) \frac{\Lambda_{uv}(x)\Upsilon_{uv}(x)}{1+\max_{x\in[0,T]} \Omega_{uv}(x)}$$

$$\precsim \mu_1(\max_{x\in[0,T]} \Gamma_{uv}(x)) \max_{x\in[0,T]} \Gamma_{uv}(x) + \mu_2(\max_{x\in[0,T]} \Gamma_{uv}(x)) \frac{\max_{x\in[0,T]} \Lambda_{uv}(x) \max_{x\in[0,T]} \Upsilon_{uv}(x)}{1+\max_{x\in[0,T]} \Omega_{uv}(x)}$$

$$\Rightarrow \max_{x\in[0,T]} \Omega_{uv}(x) \precsim \mu_1(\max_{x\in[0,T]} \Gamma_{uv}(x)) \max_{x\in[0,T]} \Gamma_{uv}(x) + \mu_2(\max_{x\in[0,T]} \Gamma_{uv}(x)) \frac{\max_{x\in[0,T]} \Lambda_{uv}(x) \max_{x\in[0,T]} \Upsilon_{uv}(x)}{1+\max_{x\in[0,T]} \Omega_{uv}(x)}$$

$\therefore d(Tu,Qv) \precsim \mu_1(d(u,v))d(u,v) + \mu_2(d(u,v)) \dfrac{d(Tu,u)d(Qv,v)}{1+d(Tu,Qv)}$ is satisfied so that the pair (T,Q) has a unique common fixed point in $X.$

Hence, the system (5.1) of linear Volterra integral equations of the second has a unique common solution.

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