International Journal of Transformation in Applied Mathematics & Statistics
Vol. 2, Issue 1 - 2019

NUMERICAL STABILITY OF FICK'S SECOND LAW TO HEAT FLOW

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ABSTRACT

In this paper we shall discuss the stability of Fick's second law to the study of heat flow using Forward Time, Centered Space or FTCS Approximation. We shall derive one dimensional heat equation from the Fick's law second with the constant of proportionality $D > 0$ called the diffusion constant given $\alpha(x,t)$ and $u(x,t)$ which represent the energy density and temperature respectively at the point x meters along a thin rod at time t (in seconds) having known substances namely a constant density ρ and specific heat C. The Fick's law is a general diffusion equation.

However, diffusion is the transport of a material or chemical by molecular motion from a region of high concentration to a region of low concentration until they are eventually uniformly distributed. We shall replace the diffusion constant D with an exponential $h^n < \frac{(\Delta x)^2}{2}$ such that $n = 1,2,3,\ldots$ and a constant $h = 0.1.$

Numerically, we shall use table to illustrate the effect of h (a fixed value) on the stability of the heat Fick's equation with the help of a working Matlab.

Keywords: Fick's Law, Heat Flow, Thermal Diffusivity, Stability, Finite Difference Approximation.

INTRODUCTION

Most models can be formulated using ordinary differential equations but more complicated problems of advanced physics and engineering involve working with partial differential equations (PDEs).

However, many PDE models involve the study of how a certain quantity changes with time and space in which conservation law plays an important rule. It is necessary to approximate the solution of these PDEs numerically in order to investigate the predictions of the mathematical models, as exact solutions are usually unavailable or rare to obtain. Fick's law of diffusion was derived by Adolf Fick in1855 which describe diffusion that can be used to solve for the diffusion coefficient, D. Fick's first law relates the diffusive flux to the concentration under the assumption of steady state.

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 ISSN: 2581-7620

It postulates that the flux goes from origins of high concentration to regions of low concentration, with a magnitude that is proportional to the concentration gradient-spactial derivative. Fick's second law predicts how diffusion causes the concentration to change with time. The second law of Fick is a partial differential equation in one dimensional.

SOME RELATED LITERATURE REVIEW

Many scientists looked at heat flows in some ways.

Abarbanel and Ditkowski (2000) studied rate of convergence of error bounds and their temporal behavior using the finite difference approximations. The result determined the dependence of the error bounds on mesh size and time.

Fukagata and Kesagi (2002) studied the cylindrical system using highly energy conservative finite difference method. The result obtained was that the energy conservation in discretized space is satisfied which holds for both equally and unequally spaced mesh on cylindrical coordinate system.

Thankane and Stys (2009) studied the effective algorithms based on finite difference method for linear and non linear beam equations. The method used was the convergence analysis of the algorithms. They obtained that the number of solution of beam equations is given by designing Mathmatica Module.

Kalyani and Ramchandra (2013) studied the one dimensional Heat equation with initial and boundary conditions using finite difference method and other numerical methods. They obtained that the solution of Heat equation was given as a polynomial of two variables by using double interpolation.

Raffaele, and Beatrice (2014) studied the numerical solution of partial differential equation (PDE) with oscillatory solutions by considering diffusion equation $u_t = du_{xx}$. They used Finite difference method which gives accurate and efficient solution of PDE- a diffusion problem with mixed boundary conditions.

On my best of knowledge, there are no more papers related to the stability of one dimensional heat equation. In order to remove the lack in the literature, we shall analyze the stability of one dimensional heat equation solved with the method of finite approximation by replacing the diffusion constant D with an exponential h^n.

DERIVATION OF HEAT EQUATION

If $\alpha(x,t)$ and $u(x,t)$ represent the energy density and temperature, then

$$\alpha(x,t) = \rho C u(x,t) \quad (1)$$

where

ρ represents density and C represents specific heat.

Applying the conversation law, we obtain

$$\rho C u_t + \tau_x = 0 \quad (2)$$

Where $\tau = \tau(x,t)$ denotes the flux of the quantity at x at time t which measures the amount of the quantity crossing section at x and at t.

Since, the heat flow follows a diffusion model,

$$\tau = -K u_x \quad (3)$$

where K represents the thermal conductivity.

Combining (2) and (3) gives

$$\rho C u_t - K u_{xx} = 0 \quad (4)$$

Equivalently,

$$\frac{\rho C u_t}{\rho C} - \frac{K u_{xx}}{\rho C} = 0.$$

$$u_t - \frac{K u_{xx}}{\rho C} = 0.$$

$$u_t - D u_{xx} = 0.$$

$$u_t = D u_{xx} \tag{5}$$

Where $D = \dfrac{K}{\rho C}$ is called the diffusivity or thermal diffusivity.

The last equation is called heat equation.

METHODS OF ANAYSIS

The forward difference in time and central difference in space approximations are used to analyze the heat equation.

The forward difference in time is given as:

$$\frac{\partial u}{\partial t}(x,t) \approx \frac{u_i^{m+1} + u_i^m}{\Delta t} \tag{6}$$

The central difference approximation in space is given as

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{\Delta t} \tag{7}$$

Numerically, the heat equation becomes:

$$\frac{u_i^{m+1} - u_i^m}{\Delta t} = \left(\frac{D}{(\Delta x)^2}\right)\left(u_{i+1}^m - 2u_i^m + u_{i-1}^m\right) \tag{8}$$

Equivalently, we have:

$$u_i^{m+1} = u_i^m + \frac{D}{(\Delta x)^2}\Delta t\left(u_{i+1}^m - 2u_i^m + u_{i-1}^m\right)$$

$$u_i^{m+1} = u_i^m + \frac{h^n}{(\Delta x)^2}\Delta t\left(u_{i+1}^m - 2u_i^m + u_{i-1}^m\right) \tag{9}$$

for $h^n < \dfrac{(\Delta x)^2}{2}$; $n = 1,2,3,\dots$

In numerical analysis, the Lax Equivalence theorem is the fundamental theorem in the analysis of finite difference methods for the numerical solution of partial differential equations. It states that for a consistent finite difference method for a well-posed linear initial value problem, the method is convergent if and only if it is stable.

NUMERICAL STABILITY OF THE GIVEN HEAT EQUATION

The table below shows some numerical values of h^n where n = 1,2,3, … with some equidistant values x and a fixed change in x; Δx=0.2.

Figure 1.A table showing numerical values involving h^n

			Δx=0.2				
n	h	D=h^n	x	x(new)	Δx.^2	D/Δx.^2	D/Δx.^2/2
1	0.1	0.1	2	2	4	0.025	0.0125
2	0.1	0.01	4	2.2	4.84	0.00206612	0.001033
3	0.1	0.001	6	2.4	5.76	0.00017361	8.68E-05
4	0.1	0.0001	8	2.6	6.76	1.4793E-05	7.4E-06
5	0.1	0.00001	10	2.8	7.84	1.2755E-06	6.38E-07
6	0.1	0.000001	12	3	9	1.1111E-07	5.56E-08
7	0.1	1E-07	14	3.2	10.24	9.7656E-09	4.88E-09
8	0.1	1E-08	16	3.4	11.56	8.6505E-10	4.33E-10
9	0.1	1E-09	18	3.6	12.96	7.716E-11	3.86E-11
10	0.1	1E-10	20	3.8	14.44	6.9252E-12	3.46E-12
11	0.1	1E-11	22	4	16	6.25E-13	3.13E-13
12	0.1	1E-12	24	4.2	17.64	5.6689E-14	2.83E-14
13	0.1	1E-13	26	4.2	17.64	5.6689E-15	2.83E-15
14	0.1	1E-14	28	4.2	17.64	5.6689E-16	2.83E-16
15	0.1	1E-15	30	4.2	17.64	5.6689E-17	2.83E-17
16	0.1	1E-16	32	4.2	17.64	5.6689E-18	2.83E-18
17	0.1	1E-17	34	4.2	17.64	5.6689E-19	2.83E-19
18	0.1	1E-18	36	4.2	17.64	5.6689E-20	2.83E-20
19	0.1	1E-19	38	4.2	17.64	5.6689E-21	2.83E-21
20	0.1	1E-20	40	4.2	17.64	5.6689E-22	2.83E-22

DISCUSSION OF RESULTS

The table above showed the condition on the diffusion constant D with an exponential $h^n < \frac{(\Delta x)^2}{2}$ such that $n = 1,2,3,\ldots$ and a constant $h = 0.1$ for which the system is stable.

CONCLUSION

The work is concluded by proving the numerical stability condition**.**

From (9):

$$u_i^{m+1} = u_i^m + k\left(\frac{\Delta t}{(\Delta x)^2}\right)\left(u_{i+1}^m - 2u_i^m + u_{i-1}^m\right)$$

$$\Rightarrow u_i^{m+1} - u_i^m = k\left(\frac{\Delta t}{(\Delta x)^2}\right)\left(u_{i+1}^m - 2u_i^m + u_{i-1}^m\right).$$

Let $u_i^m = \phi_m e^{ji\Delta x\theta}$ then $u_i^{m+1} = \phi_{m+1} e^{ji\Delta x\theta}$ and $u_{i-1}^m = \phi_m e^{j(i-1)\Delta x\theta}$

$$\Rightarrow u_i^{m+1} - u_i^m = (\phi_{m+1} - \phi_m)e^{jn\Delta x\theta}.$$

$$\therefore u_i^{m+1} - u_i^m = \frac{k\Delta t}{(\Delta x)^2}\left(\phi_m e^{j(i+1)\Delta x\theta} - 2\phi_m e^{ji\Delta x\theta} + \phi_m e^{j(i-1)\Delta x\theta}\right).$$

$$u_i^{m+1} - u_i^m = \frac{k\Delta t}{(\Delta x)^2}\left(\phi_m e^{ji\Delta x\theta + j\Delta x\theta} - 2\phi_m e^{ji\Delta x\theta} + \phi_m e^{ji\Delta x\theta - j\Delta x\theta}\right).$$

$$u_i^{m+1} - u_i^m = \frac{k\Delta t}{(\Delta x)^2}\left(e^{ji\Delta x\theta + j\Delta x\theta} - 2e^{ji\Delta x\theta} + e^{ji\Delta x\theta - j\Delta x\theta}\right)\phi_m.$$

$$u_i^{m+1} - u_i^m = \frac{k\Delta t}{(\Delta x)^2}\left(e^{ji\Delta x\theta}.e^{j\Delta x\theta} - 2e^{ji\Delta x\theta} + e^{ji\Delta x\theta}.e^{-j\Delta x\theta}\right)\phi_m$$

$$u_i^{m+1} - u_i^m = \frac{k\Delta t}{(\Delta x)^2}\left(e^{j\Delta x\theta} - 2 + e^{-j\Delta x\theta}\right)e^{ji\Delta x\theta}\phi_m.$$

$$u_i^{m+1} - u_i^m = \frac{k\Delta t}{(\Delta x)^2}\left(e^{j\Delta x\theta} + e^{-j\Delta x\theta} - 2\right)e^{ji\Delta x\theta}\phi_m.$$

$$u_i^{m+1} - u_i^m = \frac{k\Delta t}{(\Delta x)^2}\left(2\cos(\theta\Delta x) - 2\right)\phi_m e^{ji\Delta x\theta}.$$

$$u_i^{m+1} - u_i^m = \frac{k\Delta t}{(\Delta x)^2}\left(e^{ji\Delta x\theta} + e^{-j\Delta x\theta} - 2\right)e^{ji\Delta x\theta}\phi_m.$$

$$u_i^{m+1} - u_i^m = \frac{2k\Delta t}{(\Delta x)^2}\left(\cos(\theta\Delta x) - 1\right)\phi_m e^{ji\Delta x\theta}.$$

$$u_i^{m+1} = u_i^m + \frac{2k\Delta t}{(\Delta x)^2}\left(\cos(\theta\Delta x) - 1\right)\phi_m e^{ji\Delta x\theta}.$$

$$\phi_{\mathrm{m}+1} e^{ji\Delta x\theta} = \phi_m e^{ji\Delta x\theta} + \frac{2k\Delta t}{(\Delta x)^2}\left(\cos(\theta\Delta x) - 1\right)\phi_m e^{ji\Delta x\theta}.$$

$$\phi_{m+1} e^{ji\Delta x\theta} = \left(\phi_m + \frac{2k\Delta t}{(\Delta x)^2}\left(\cos\theta\Delta x - 1\right)\phi_m\right)e^{ji\Delta x\theta}.$$

$$\phi_{m+1} = \left(1 + \frac{2k\Delta t}{(\Delta x)^2}\left(\cos\theta\Delta x - 1\right)\right)\phi_m.$$

$$\phi_{m+1} = \left(1 + \frac{2k\Delta t}{(\Delta x)^2}\left(-2\sin^2\left(\frac{\theta\Delta x}{2}\right)\right)\right)\phi_m.$$

$$\phi_{m+1} = \left(1 - \frac{4k\Delta t}{(\Delta x)^2}\sin^2\left(\frac{\theta\Delta t}{(\Delta x)^2}\right)\right)\phi_m.$$

Now, for stability, it is required that $\phi_m \le \phi_{m+1}$ then $|\phi_{m+1}| \le |\phi_m|$ so that $|\phi_{m+1}| = |1 - \frac{4k\Delta t}{(\Delta x)^2}\sin^2\left(\frac{\theta\Delta x}{2}\right)|.|\phi_m|$.

$$\frac{|\phi_{m+1}|}{|\phi_m|} = |1 - \frac{4k\Delta t}{(\Delta x)^2}\sin^2\left(\frac{\theta\Delta x}{2}\right)| \le 1.$$

Since $\frac{|\phi_{m+1}|}{|\phi_m|} \le 1$.

$$\Rightarrow |1 - \frac{4k\Delta t}{(\Delta x)^2}\sin^2\left(\frac{\theta\Delta x}{2}\right)| \le 1.$$

$$\Leftrightarrow -1 \le 1 - \frac{4k\Delta t}{(\Delta x)^2}\sin^2\left(\frac{\theta\Delta x}{2}\right) \le 1.$$

$$\therefore -2 \le -\frac{4k\Delta t}{(\Delta x)^2}\sin^2\left(\frac{\theta\Delta x}{2}\right) \le 0.$$

The right inequality is satisfied automatically, while the left inequality can be re-written in the form: $\frac{4k}{(\Delta x)^2}\Delta t \sin^2\left(\frac{\theta \Delta x}{2}\right) \leq 2.$

$$\Rightarrow \frac{k}{(\Delta x)^2}\Delta t \sin^2\left(\frac{\theta \Delta x}{2}\right) \leq \frac{1}{2}.$$

$\therefore \frac{k}{(\Delta x)^2} \leq \frac{1}{2}$ provided that $\Delta t \sin^2\left(\frac{\theta \Delta x}{2}\right) \leq \Delta t \Rightarrow \sin^2\left(\frac{\theta \Delta x}{2}\right) \leq 1$.

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